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Speciální funkce a transformace ve zpracování obrazu

Upozornění

Tento soubor obsahuje veškeré slajdy k předmětu SFTO, část „Momenty“, kromě některých reálných experimentů prezentovaných na přednášce. Je určen výhradně ke studijním účelům. Jakékoliv jiné použití, umísťování na web nebo jiné šíření souboru nebo jednotlivých snímků je možné jen s výslovným souhlasem autorů.

Slajdy nejsou samostatným studijním materiálem, nenahrazují přednášku a nepokrývají všechny požadavky ke zkoušce.

Důležité informace

- Rozsah: LS, 2+0, Zk
- http://zoi.utia.cz/teaching
- Moment homepage
  http://staff.utia.cas.cz/zitova/tutorial/icip07.html

Předcházející předměty

- ROZ 1, 2 (FJFI)

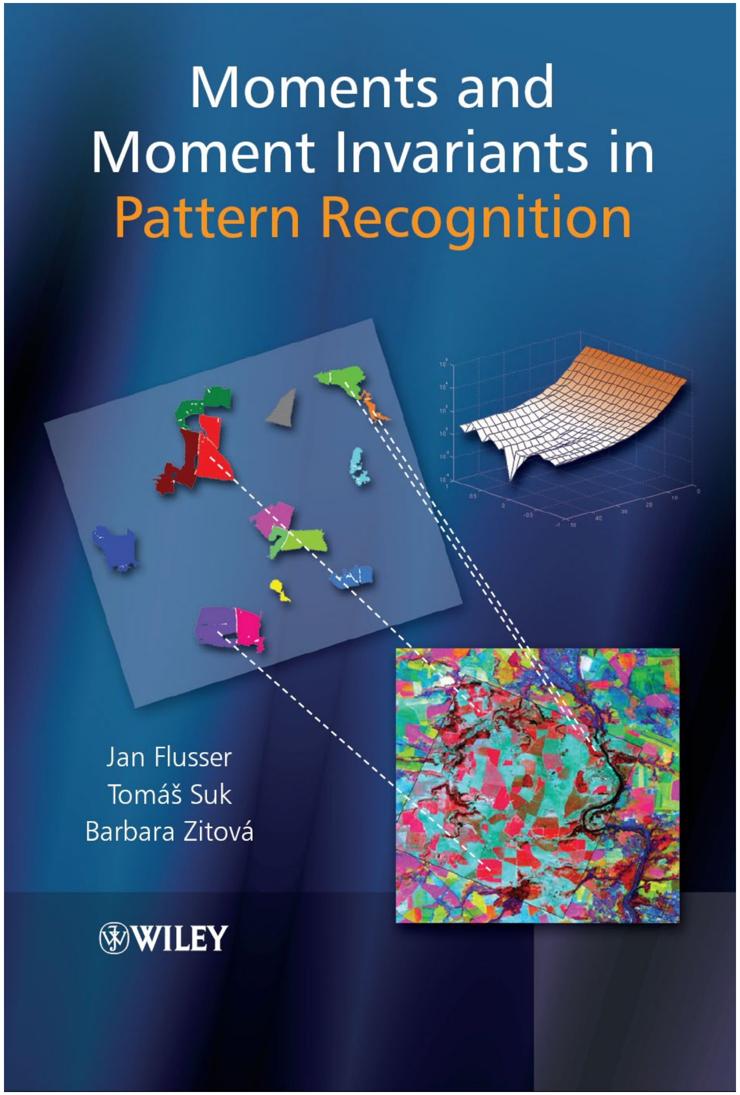
- DZO (MFF)

Souběžné a navazující předměty

- Speciální seminář NPGR022 (MFF)
- Variační metody NPGR029 (MFF)

Hlavní části

- **Waveletová transformace (B. Zitová)**
- **Momentové funkce (J. Flusser)**



The Textbook

Wiley, London, 2009

http://zoi.utia.cas.cz/moment_invariants

Object recognition

Recognition (classification) = assigning a pattern/object to one of pre-defined classes

The object is described by its features

Features – measurable quantities, usually form an *n*-D vector in a metric space

Problem formulation

Non-ideal imaging conditions → degradation of the image

$$g = D(f)$$

D - unknown degradation operator

Basic approaches

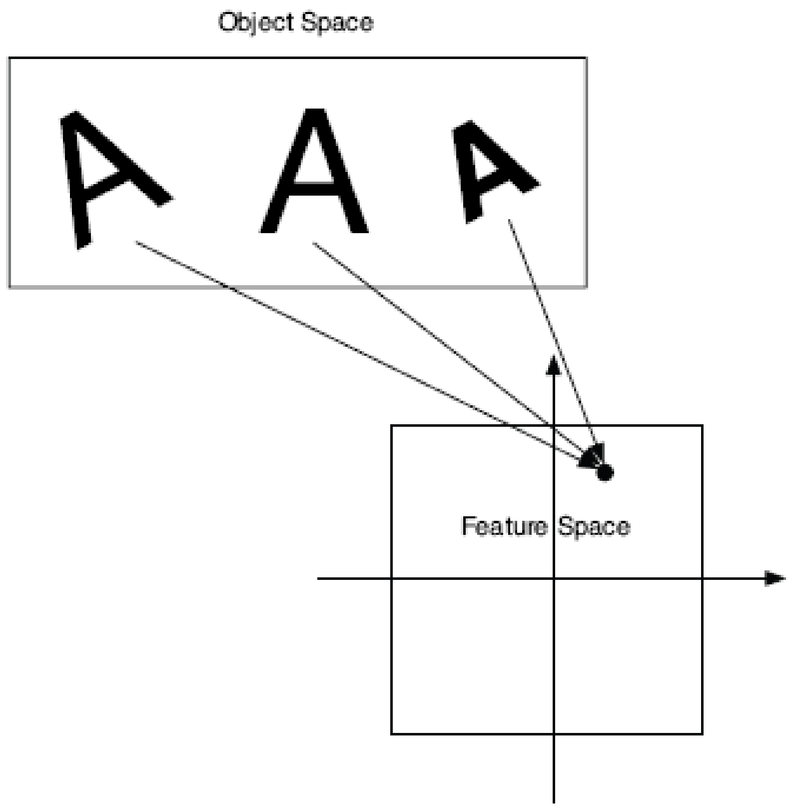
- Brute force
- Normalized position → inverse problem
- Description of the objects by invariants

What are invariants?

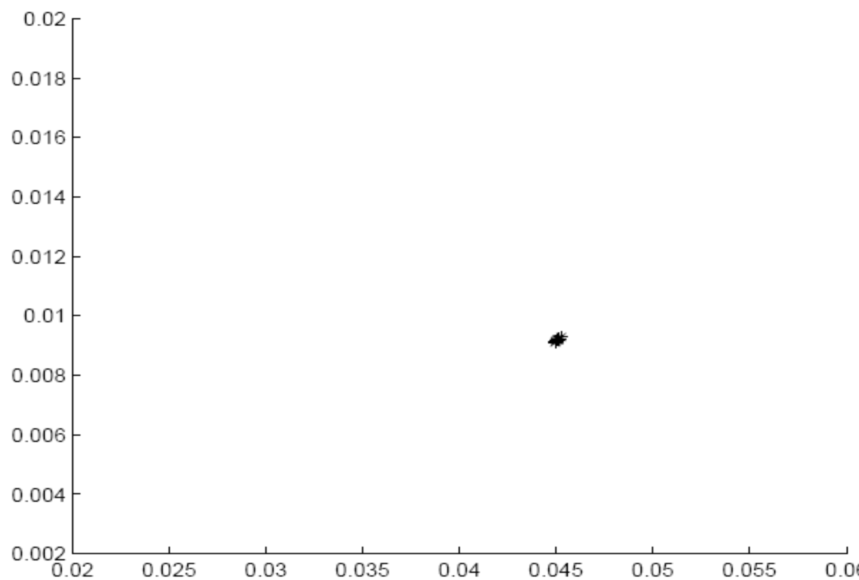
Invariants are functionals defined on the image space such that

- $I(f) = I(D(f))$ for all admissible D

What are invariants?



Example: TRS

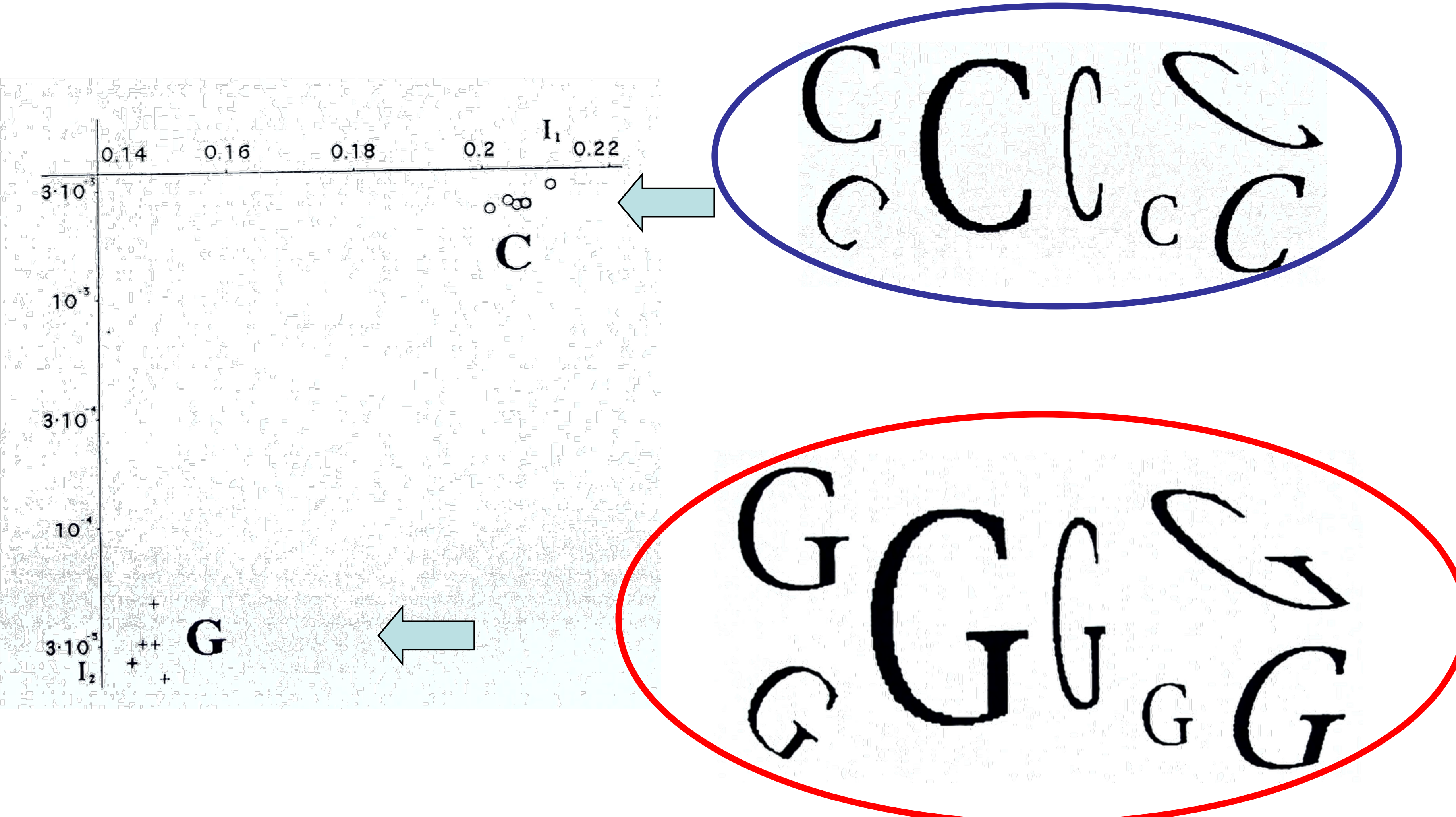

0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0.02
0.025
0.03
0.035
0.04
0.045
0.05
0.055
0.06

What are invariants?

Invariants are functionals defined on the image space such that

- $I(f) = I(D(f))$ for all admissible D
- $I(f_1), I(f_2)$ "different enough" for different f_1, f_2

Discrimination power



Major categories of invariants

Simple shape descriptors

- compactness, convexity, elongation, ...

Transform coefficient invariants

- Fourier descriptors, wavelet features, ...

Point set invariants

- positions of dominant points

Differential invariants

- derivatives of the boundary

Moment invariants

What are moment invariants?

Functions of image moments, invariant to certain class of image degradations

- Rotation, translation, scaling
- Affine transform
- Elastic deformations
- Convolution/blurring
- Combined invariants

What are moments?

Moments are "projections" of the image function into a polynomial basis

$f(x, y)$ – piecewise continuous image function defined on bounded $\Omega \subset \mathcal{R} \times \mathcal{R}$

$\{\mathcal{P}_{pq}(x, y)\}$ – set of polynomials defined on Ω

$$M_{pq} = \int\int \mathcal{P}_{pq}(x, y) f(x, y) dx dy$$

What are moments?

Moments are not " linear-algebraic coordinates" of the image in the polynomial basis

$$f(x, y) \neq \sum M_{pq} \mathcal{P}_{pq}(x, y)$$

The most common moments

Geometric moments

$$m_{pq}^{(f)} = \int_{R^2} x^p y^q f(x,y)\,dx\,dy$$

$(p + q)$ - the order of the moment

Geometric moments – the meaning

$$m_{pq}^{(f)} = \int_{R^2} x^p y^q f(x,y) dx dy$$

0^{th} order - area

1^{st} order - center of gravity

$$x_t = \frac{m_{10}}{m_{00}}, \qquad y_t = \frac{m_{01}}{m_{00}}$$

2^{nd} order - moments of inertia

3^{rd} order - skewness

Geometric moments

$$m_{pq}^{(f)} = \int_{R^2} x^p y^q f(x,y) dx dy$$

Uniqueness theorem

If $f(x,y)$ is piecewise continuous and Ω is bounded then

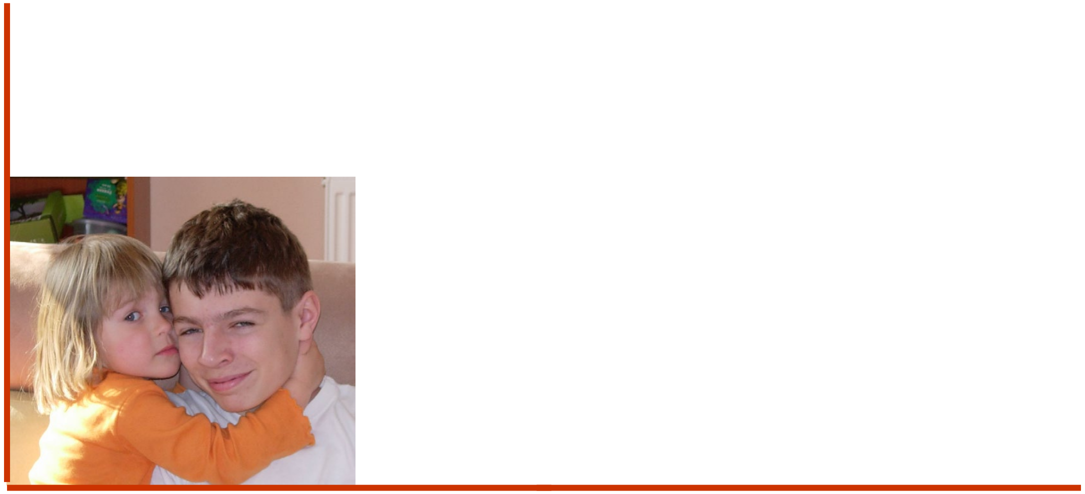
$$f(x,y) \Longleftrightarrow \{m_{pq}\} \quad p,q = 0,1,2,\cdots,\infty$$

Invariants to translation

Central moments

$$\mu_{pq}^{(f)} = \int_{R^2} (x - x_t)^p (y - y_t)^q f(x, y) dx dy$$

$$x_t = m_{10}/m_{00}, \quad y_t = m_{01}/m_{00}$$



$$x' = x + a$$
$$y' = y + b$$

Invariants to translation

Central moments

$$\mu_{pq}^{(f)} = \int_{R^2} (x - x_t)^p (y - y_t)^q f(x,y)\,dx\,dy$$

$$x_t = m_{10}/m_{00}, \quad y_t = m_{01}/m_{00}$$

$$\mu_{pq} = \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} (-1)^{k+j} x_t^k y_t^j m_{p-k,q-j}$$

Invariants to translation and scaling

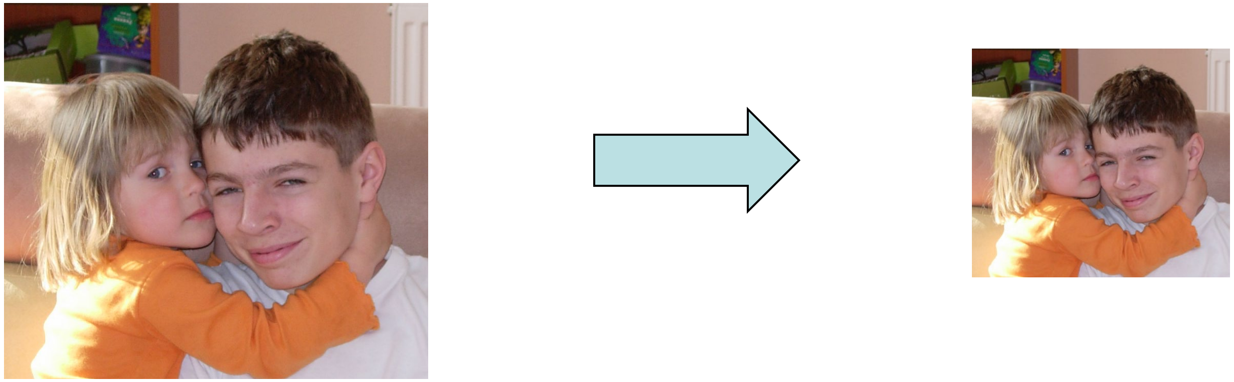
$$x' = s \cdot x + a$$

$$y' = s \cdot y + b$$

Invariants to translation and scaling

Normalized central moments

$$\nu_{pq} = \frac{\mu_{pq}}{\mu_{00}^{w}} \qquad w = \frac{p+q}{2} + 1$$



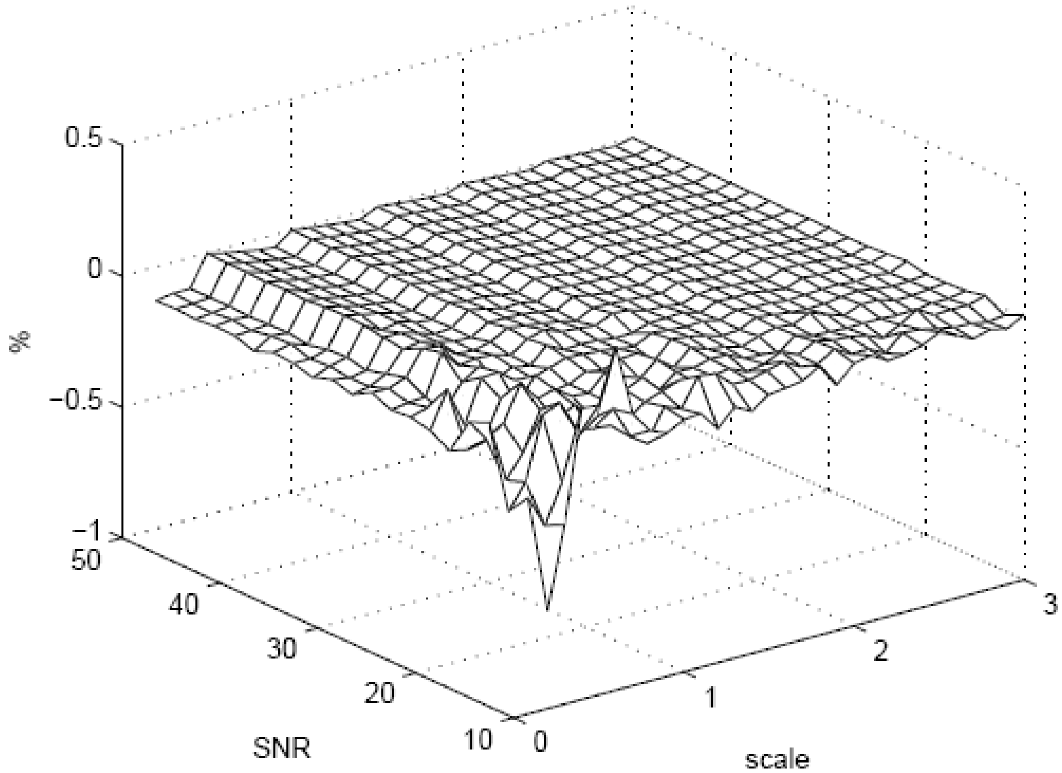
Invariants to translation and scaling

Normalized central moments

$$\nu_{pq} = \frac{\mu_{pq}}{\mu_{00}^{w}} \qquad w = \frac{p+q}{2} + 1$$

Invariants to translation and scaling

Normalized central moments



Invariants to rotation

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

Invariants to rotation

M.K. Hu, 1962 - 7 invariants of the 3rd order

$$\phi_1 = \mu_{20} + \mu_{02}$$

$$\phi_2 = (\mu_{20} - \mu_{02})^2 + 4\mu_{11}^2$$

$$\phi_3 = (\mu_{30} - 3\mu_{12})^2 + (3\mu_{21} - \mu_{03})^2$$

$$\phi_4 = (\mu_{30} + \mu_{12})^2 + (\mu_{21} + \mu_{03})^2$$

$$
\begin{aligned}
\phi_5 &= (\mu_{30} - 3\mu_{12})(\mu_{30} + \mu_{12})((\mu_{30} + \mu_{12})^2 - 3(\mu_{21} + \mu_{03})^2) \\
&+ (3\mu_{21} - \mu_{03})(\mu_{21} + \mu_{03})(3(\mu_{30} + \mu_{12})^2 - (\mu_{21} + \mu_{03})^2)
\end{aligned}
$$

$$
\begin{aligned}
\phi_6 &= (\mu_{20} - \mu_{02})((\mu_{30} + \mu_{12})^2 - (\mu_{21} + \mu_{03})^2) + \\
&\quad 4\mu_{11}(\mu_{30} + \mu_{12})(\mu_{21} + \mu_{03})
\end{aligned}
$$

$$
\begin{aligned}
\phi_7 &= (3\mu_{21} - \mu_{03})(\mu_{30} + \mu_{12})((\mu_{30} + \mu_{12})^2 - 3(\mu_{21} + \mu_{03})^2) \\
&- (\mu_{30} - 3\mu_{12})(\mu_{21} + \mu_{03})(3(\mu_{30} + \mu_{12})^2 - (\mu_{21} + \mu_{03})^2)
\end{aligned}
$$

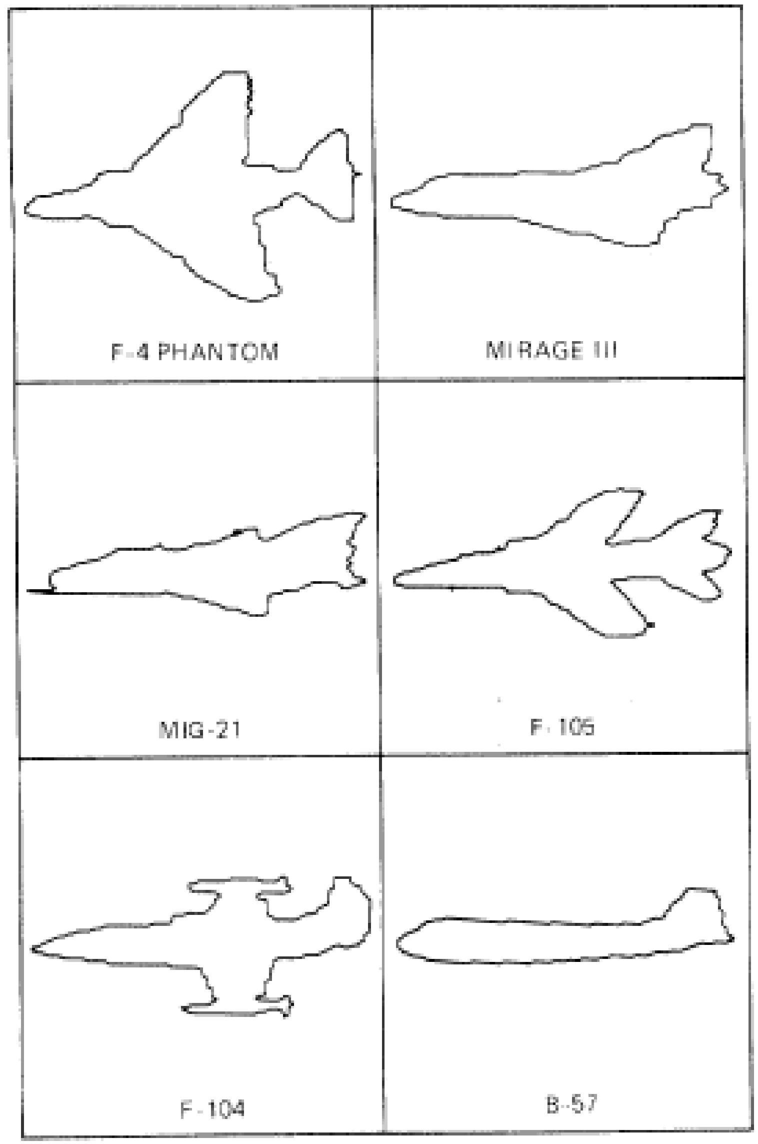
Hard to find, easy to prove:

$$\mu'_{20} = \cos^2\theta \cdot \mu_{20} + \sin^2\theta \cdot \mu_{02} - \sin 2\theta \cdot \mu_{11}$$

$$\mu'_{02} = \sin^2\theta \cdot \mu_{20} + \cos^2\theta \cdot \mu_{02} + \sin 2\theta \cdot \mu_{11}$$

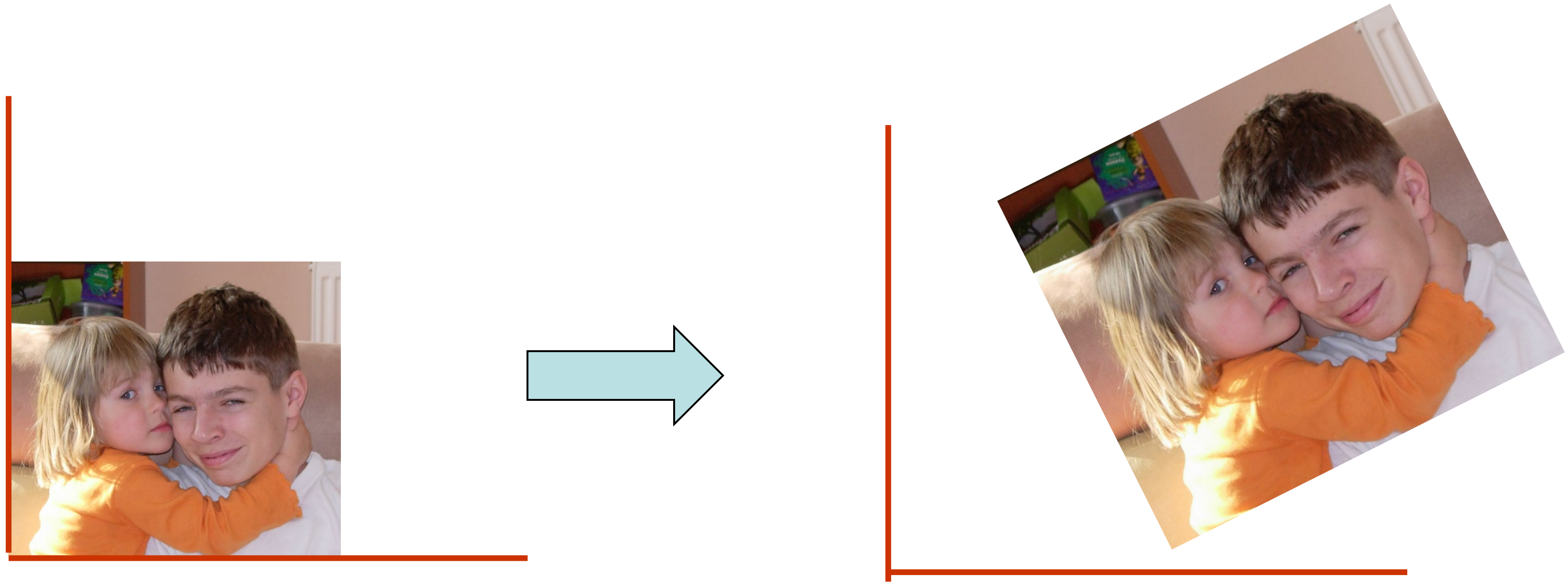
$$\mu'_{11} = \frac{1}{2}\sin 2\theta \cdot (\mu_{20} - \mu_{02}) + \cos 2\theta \cdot \mu_{11}$$

Aircraft recognition (Dudani et al., 1977)



Invariants to TRS

In any rotation invariant use the normalized moments instead of central ones



Drawbacks of the Hu's invariants

Dependence

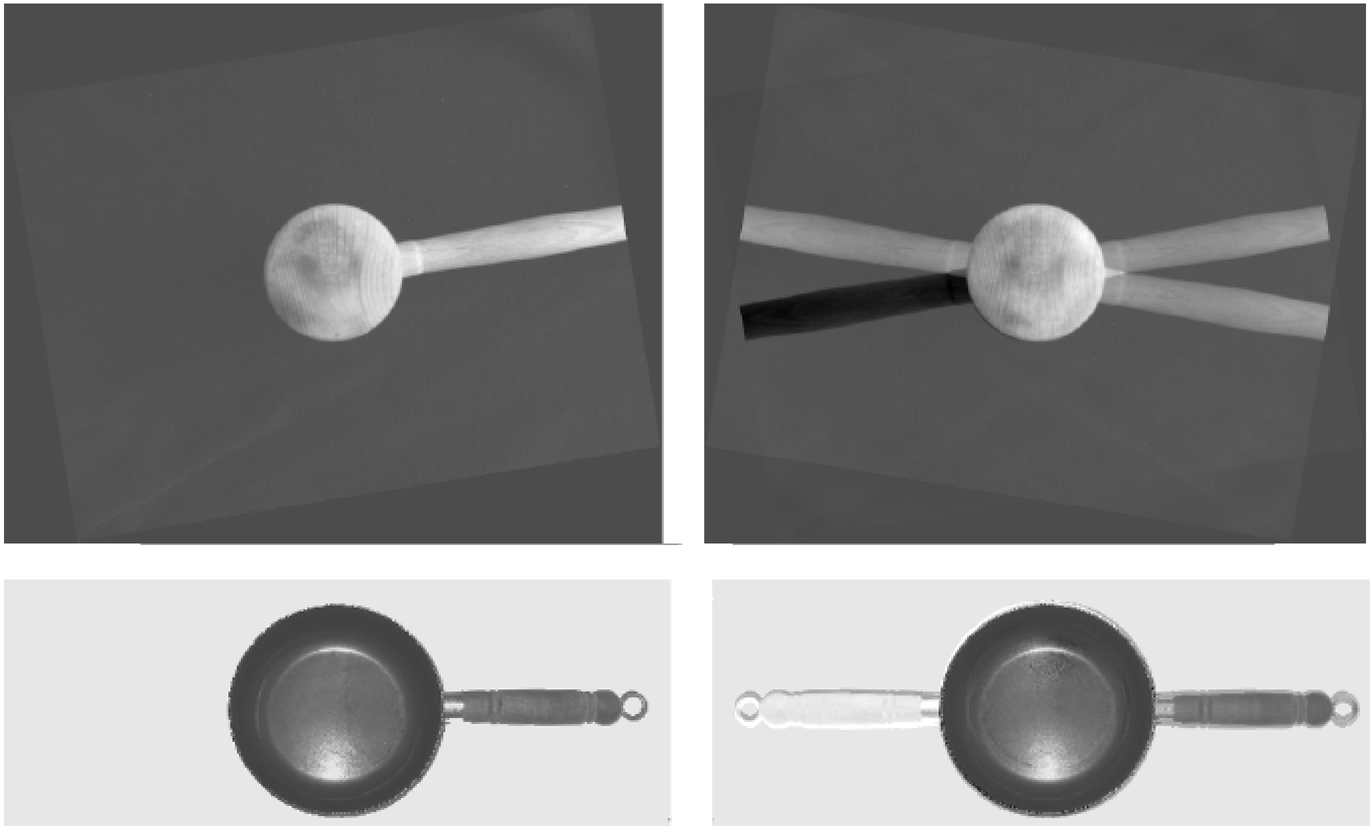
$$\phi_3 = \frac{\phi_5^2 + \phi_7^2}{\phi_4^3}$$

Incompleteness

$$m_{11}^2 = \frac{1}{4}(\phi_2 - (\frac{\phi_6}{\phi_4})^2)$$

Insufficient number → low discriminability

Consequence of the incompleteness of the Hu's set



The images not distinguishable by the Hu's set

General construction of rotation invariants

Complex moment

$$c_{pq}^{(f)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+iy)^p (x-iy)^q f(x,y)\,dx\,dy$$

Basic relations between the moments

$$c_{pq} = \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} (-1)^{q-j} \cdot i^{p+q-k-j} \cdot m_{k+j,p+q-k-j}$$

$$m_{pq} = \frac{1}{2^{p+q} i^{q}} \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} (-1)^{q-j} \cdot c_{k+j,p+q-k-j}$$

$$c_{qp} = c_{pq}^{*}$$

Examples

$$c_{00} = m_{00}$$
$$c_{10} = m_{10} + im_{01}$$
$$c_{20} = m_{20} - m_{02} + 2im_{11}$$
$$c_{11} = m_{20} + m_{02}$$

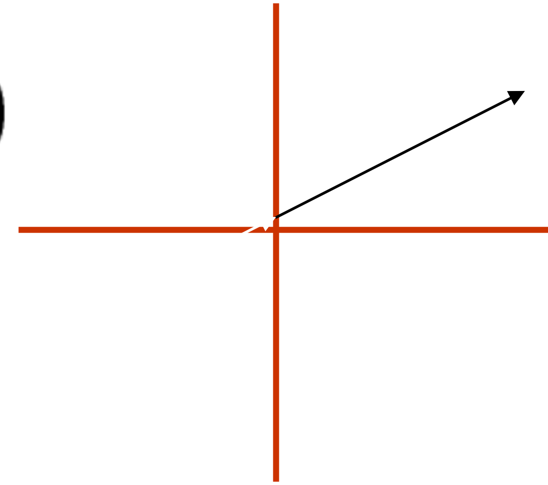
Complex moments in polar coordinates

$$c_{pq}^{(f)} = \int_0^{\infty} \int_0^{2\pi} r^{p+q+1} e^{i(p-q)\theta} f(r,\theta) d\theta dr$$

Rotation property of complex moments

$$f'(r,\theta) = f(r,\theta+\alpha)$$

$$c'_{pq} = e^{-i(p-q)\alpha} \cdot c_{pq}$$



The magnitude is preserved, the phase is shifted by *$(p-q)\alpha$.*

Invariants are constructed by a proper phase cancellation

Rotation invariants from complex moments

$$I = \prod_{i=1}^{n} c_{p_i q_i}^{k_i} \qquad \sum_{i=1}^{n} k_i (p_i - q_i) = 0$$

Examples:

$$c_{11}, c_{20} \cdot c_{02}, c_{20} \cdot c_{12}^2, \cdots, c_{pp}, c_{pq} \cdot c_{qp}, \cdots$$

How to select a complete and independent subset (basis) of the rotation invariants?

Construction of the basis

$$\forall p, q : \qquad \Phi(p,q) \equiv c_{pq} c_{q_0 p_0}^{p-q}$$

$$\begin{aligned} & p + q \leq r \\ & p \geq q \\ & p_0 + q_0 \leq r \\ & p_0 - q_0 = 1 \\ & c_{p_0 q_0} \neq 0 \end{aligned}$$

This is the basis of invariants up to the order *r*

The basis of the 3rd order

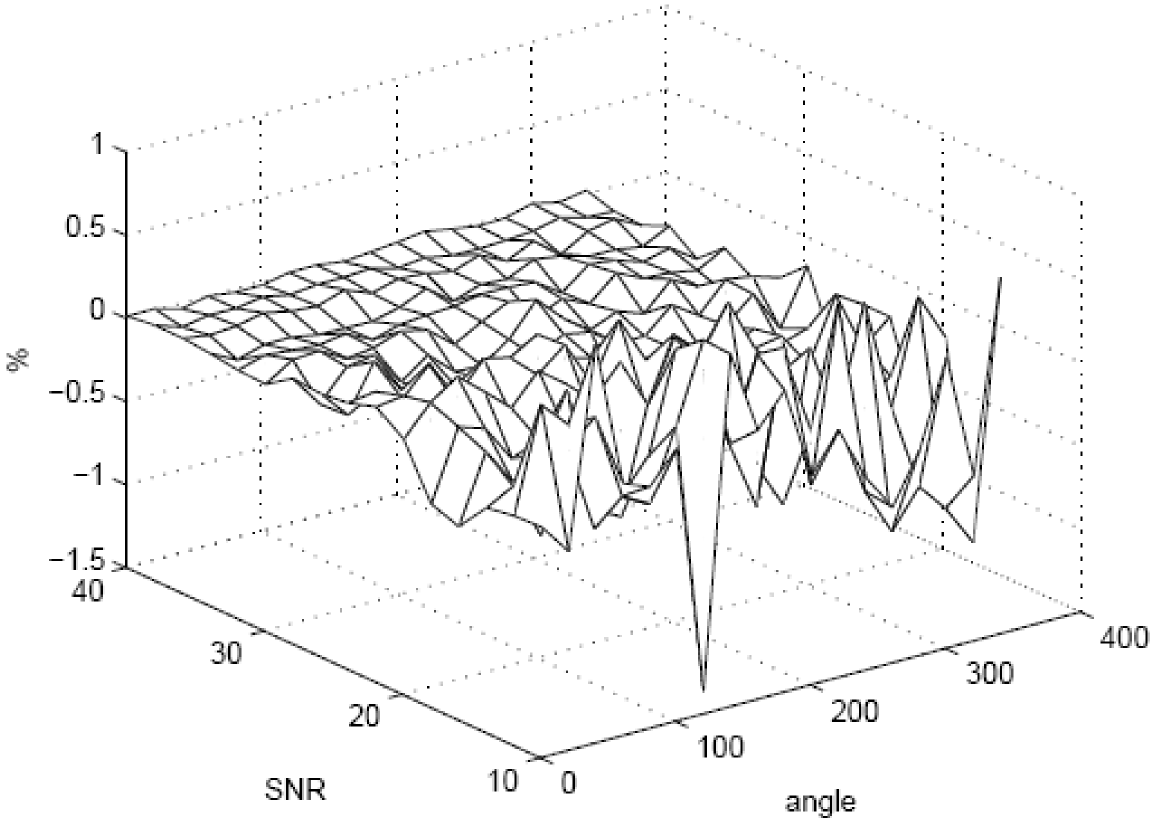
$p_0 = 2,\ q_0 = 1$

$$
\begin{aligned}
\Phi(1,1) &= c_{11} \\
\Phi(2,1) &= c_{21}c_{12} \\
\Phi(2,0) &= c_{20}c_{12}^2 \\
\Phi(3,0) &= c_{30}c_{12}^3
\end{aligned}
$$

This is basis B_3 (contains six real elements)

Numerical properties

$$\Phi(2,0) = c_{20}c_{12}^2$$



Comparing B_3 to the Hu's set

$$
\begin{aligned}
\phi_1 &= c_{11} \\
\phi_2 &= c_{20}c_{02} \\
\phi_3 &= c_{30}c_{03} \\
\phi_4 &= c_{21}c_{12} \\
\phi_5 &= \mathrm{Re}(c_{30}c_{12}^3) \\
\phi_6 &= \mathrm{Re}(c_{20}c_{12}^2) \\
\phi_7 &= \mathrm{Im}(c_{30}c_{12}^3)
\end{aligned}
$$

Drawbacks of the Hu's invariants are evident now

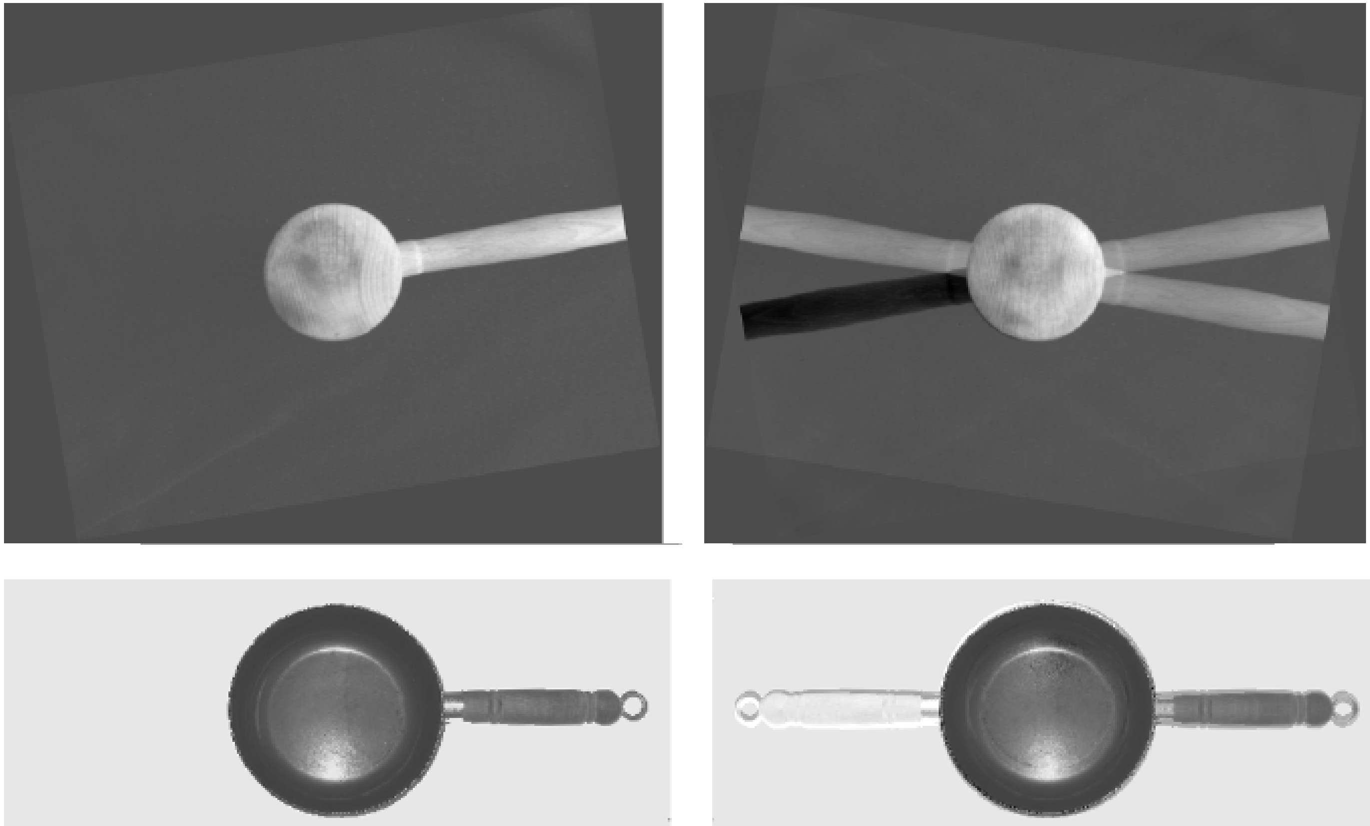
Dependence

$$\phi_3 = \frac{\phi_5^2 + \phi_7^2}{\phi_4^3}$$

Incompleteness

$$m_{11}^2 = \frac{1}{4}(\phi_2 - (\frac{\phi_6}{\phi_4})^2)$$

Comparing B_3 to the Hu's set - Experiment



The images distinguishable by B_3 but not by Hu's set

Inverse problem

$$
\begin{aligned}
\Phi(p_0, q_0) &= c_{p_0 q_0} c_{q_0 p_0} \\
\Phi(0,0) &= c_{00} \\
\Phi(1,0) &= c_{10} c_{q_0 p_0} \\
\Phi(2,0) &= c_{20} c_{q_0 p_0}^2 \\
\Phi(1,1) &= c_{11} \\
\Phi(3,0) &= c_{30} c_{q_0 p_0}^3 \\
&\cdots \\
\Phi(r,0) &= c_{r0} c_{q_0 p_0}^r \\
\Phi(r-1,1) &= c_{r-1,1} c_{q_0 p_0}^{r-2} \\
&\cdots
\end{aligned}
$$

Is it possible to resolve this system ?

Solution to the inverse problem

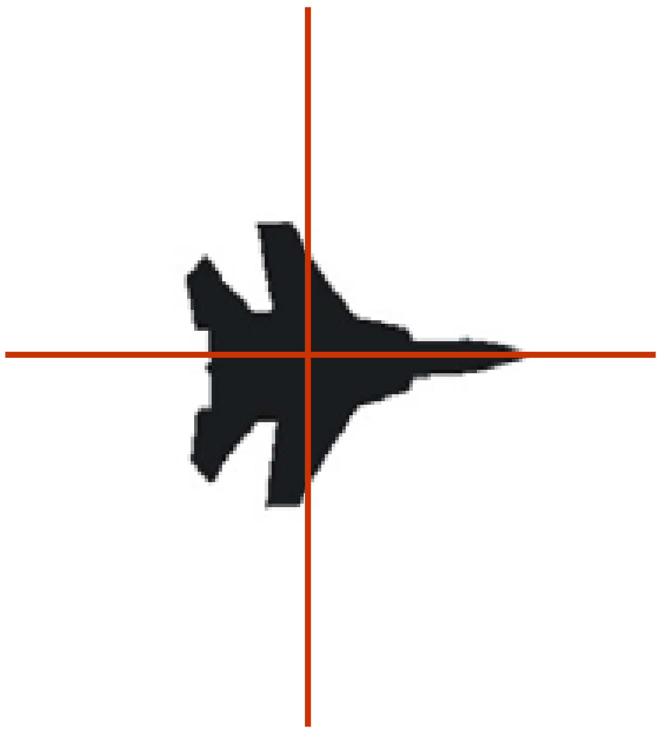
The image orientation cannot be recovered in principle → one degree of freedom → let us constrain $c_{p_0 q_0}$ to be **real and positive.** Then

$$c_{p_0 q_0} = \sqrt{\Phi(p_0, q_0)}$$

$$c_{pq} = \frac{\Phi(p, q)}{c_{q_0 p_0}^{p-q}}$$

TSR normalization using moments

Constraints on certain moments



Normalized position to scaling

Constraint $m'_{00} = 1$

Normalized position to translation

Constraints

$$m'_{10} = 0$$
$$m'_{01} = 0$$

Normalized position to rotation

Constraint

$$\mu'_{11} = 0$$

Rotation angle

$$\tan 2\theta = \frac{-2\mu_{11}}{\mu_{20} - \mu_{02}}$$

Removing ambiguity

Additional constraints

$$\mu'_{20} \geq \mu'_{02}$$
$$\mu'_{30} \geq 0$$

Moments after normalization

$$\mu'_{20} \equiv \lambda_1 = [(\mu_{20}+\mu_{02})+\sqrt{(\mu_{20}-\mu_{02})^2+4\mu_{11}^2}]/2$$

$$\mu'_{02} \equiv \lambda_2 = [(\mu_{20}+\mu_{02})-\sqrt{(\mu_{20}-\mu_{02})^2+4\mu_{11}^2}]/2$$

The moments of the normalized image are invariants. Compare to the Hu's set !

Moments after normalization

$$\mu'_{20} = (\phi_1 + \sqrt{\phi_2})/2$$

$$\mu'_{02} = (\phi_1 - \sqrt{\phi_2})/2$$

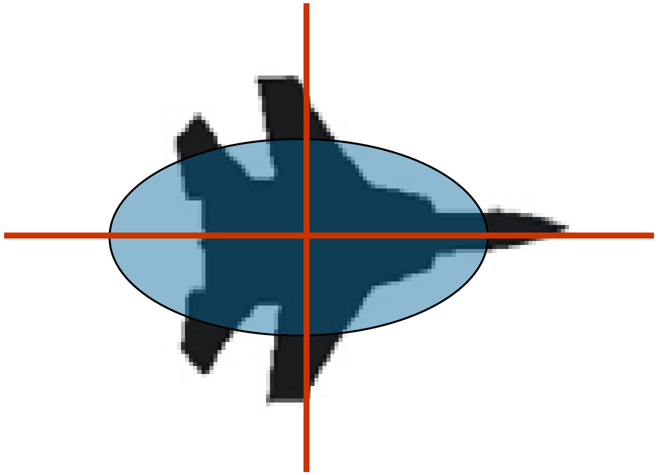
An alternative approach to constructing invariants

Reference ellipse

An ellipse having the same 2nd order moments as the original object

$$\mu'_{20} = \frac{\pi a^3 b}{4}$$

$$\mu'_{02} = \frac{\pi a b^3}{4}$$



a, b – major/minor semiaxis

Normalization as an eigenproblem

$$M = \begin{pmatrix} \mu_{20} & \mu_{11} \\ \mu_{11} & \mu_{02} \end{pmatrix}$$

$$M' = G^T M G = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \mu'_{20} & 0 \\ 0 & \mu'_{02} \end{pmatrix}$$

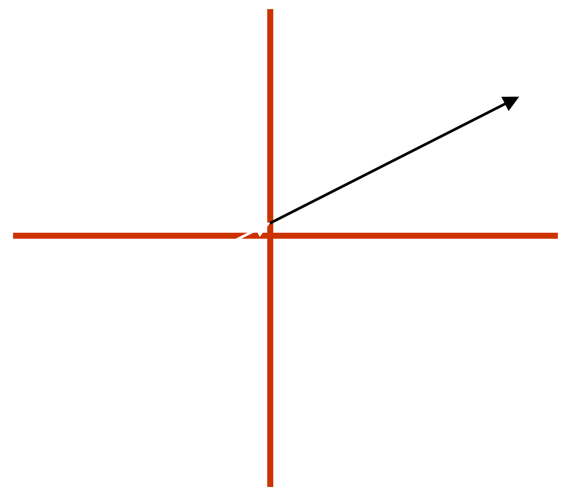
$$|M - \lambda I| = 0$$

Normalization by complex moments

Constraint

$$c_{st} \text{ real and positive} \Longrightarrow$$

$$\alpha = \frac{1}{s-t} \cdot \arctan\left(\frac{\mathrm{Im}\, c_{st}}{\mathrm{Re}\, c_{st}}\right)$$



Normalization by complex moments

Constraint

$$c_{st} \text{ real and positive} \Longrightarrow$$

$$\alpha = \frac{1}{s-t} \cdot \arctan\left(\frac{\mathrm{Im} c_{st}}{\mathrm{Re} c_{st}}\right)$$

If *s = 2, t = 0* → traditional normalization

$$c_{20} = m_{20} - m_{02} + 2im_{11}$$

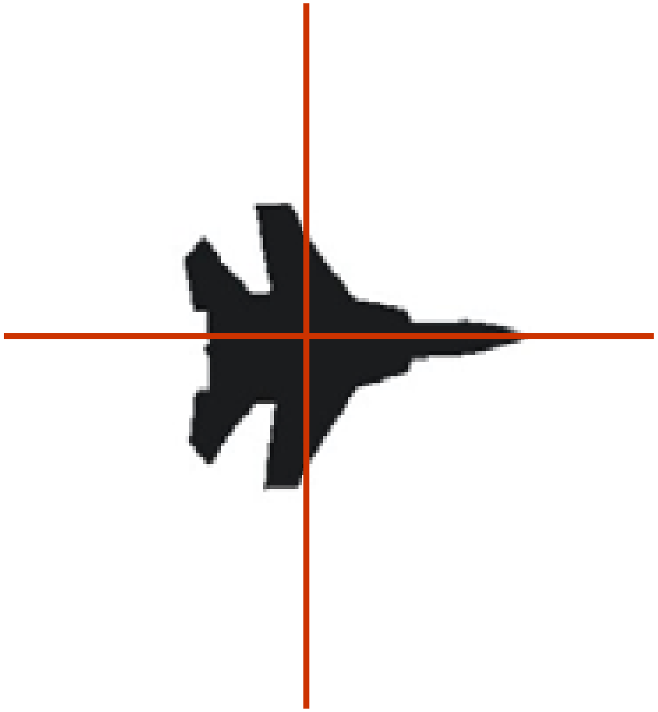
Normalization by complex moments

Constraint

$$c_{st} \text{ real and positive} \Longrightarrow$$

Unambiguous only if $s - t = 1$

An example: $s = 6$, $t = 0$



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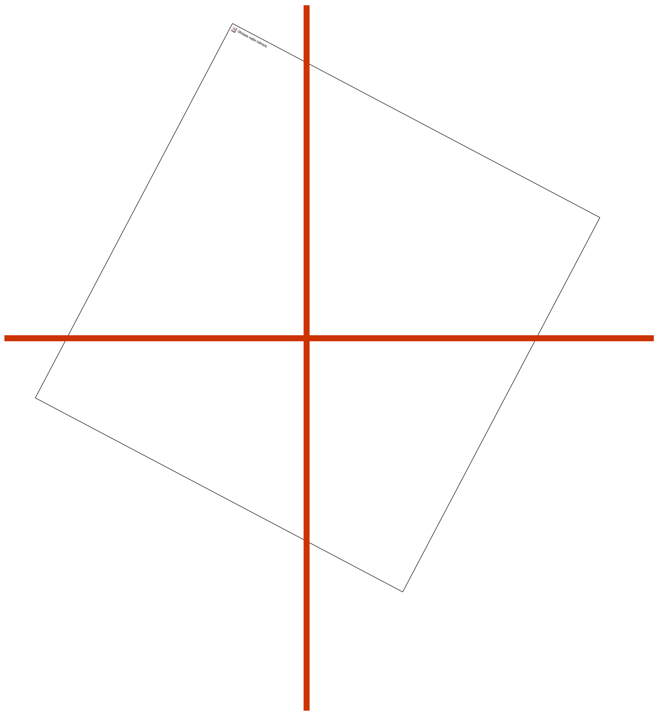

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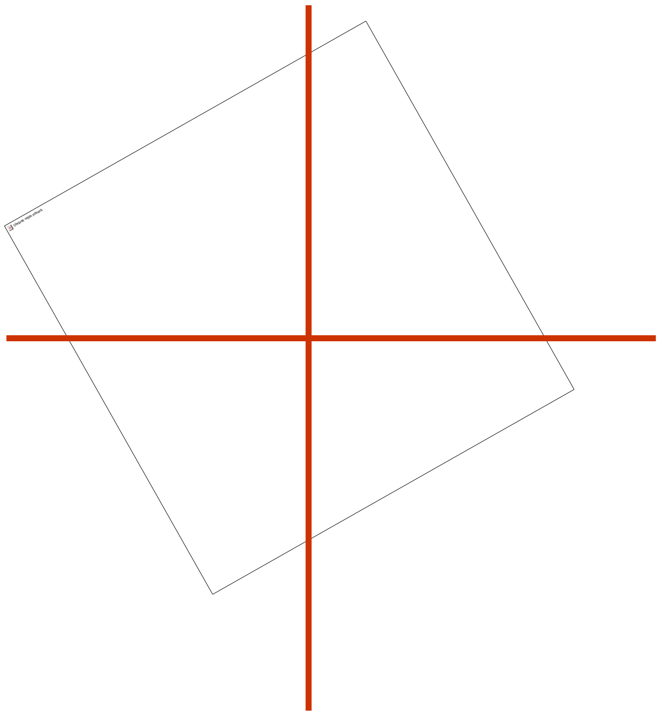


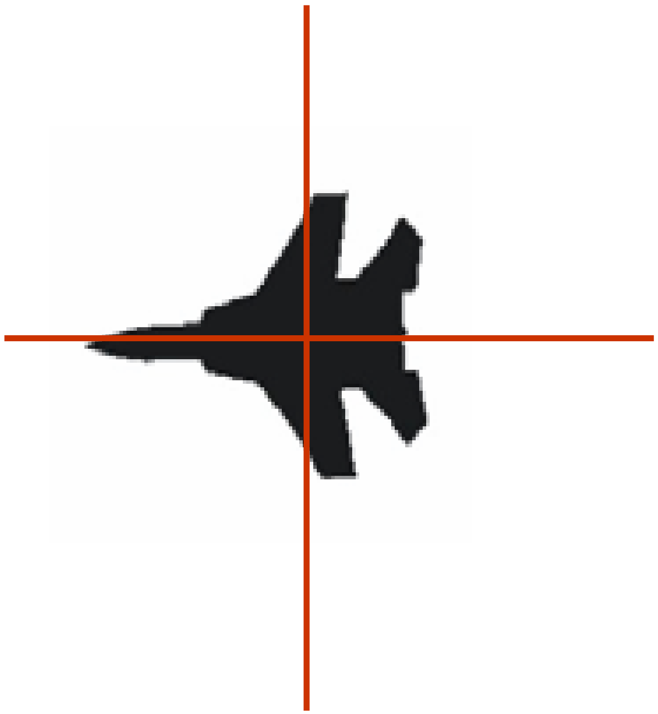
Normalization by complex moments

Constraint

$$c_{st} \text{ real and positive} \Longrightarrow$$

Unambiguous only if *s* - *t* = 1

An example: *s* = 6, *t* = 0



Normalization by complex moments

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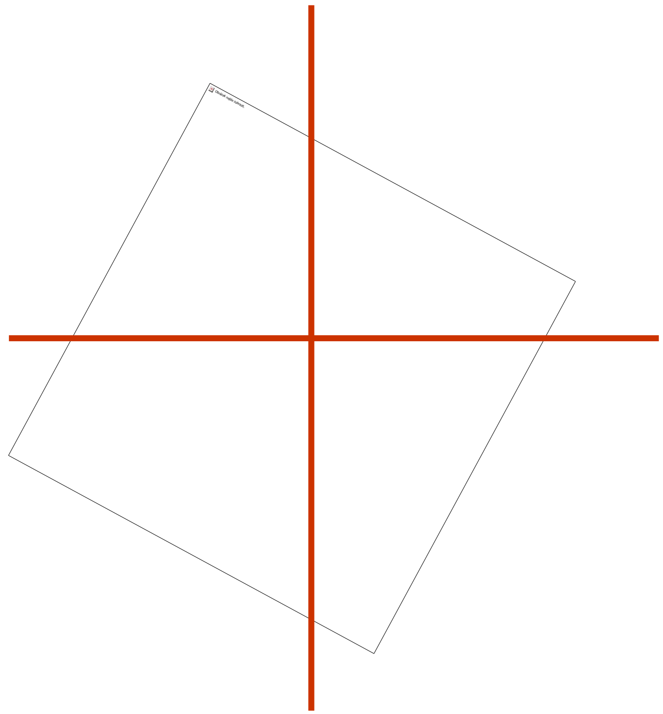

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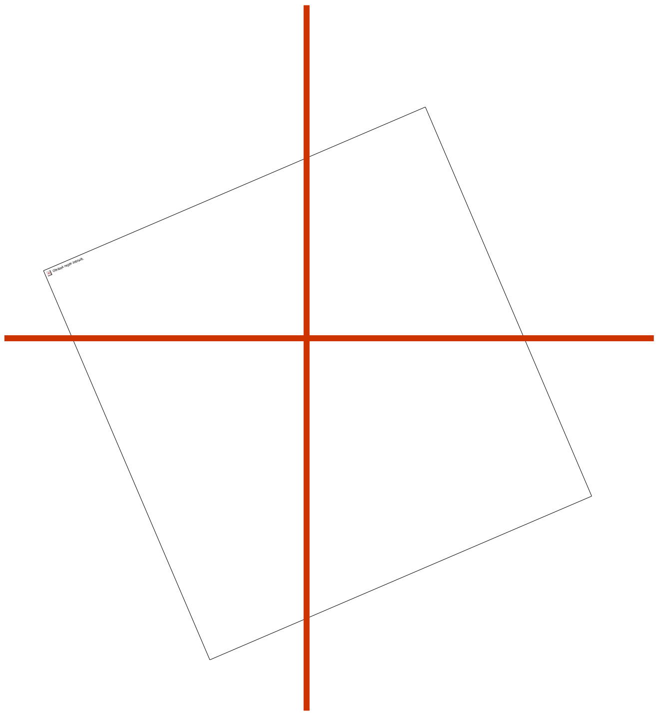


Rotation invariants via normalization

- Theoretically equivalent to the previous construction
- Leads to different basis (more complicated formulas)
- No need to actually transform the object

Pseudoinvariants

How do the rotation invariants behave under mirror reflection?

$$\overline{f}(x,y) = f(x,-y)$$

Pseudoinvariants

How do the rotation invariants behave under mirror reflection?

$$\overline{f}(x,y) = f(x,-y)$$

Pseudoinvariants

How do the rotation invariants behave under mirror reflection?

$$\overline{f}(x,y) = f(x,-y)$$

$$\overline{c_{pq}} = c_{pq}^*$$

$$\overline{\Phi(p,q)} = \overline{c_{pq}c_{q_0p_0}^{p-q}} = c_{pq}^* \cdot (c_{q_0p_0}^*)^{p-q} = \Phi(p,q)^*$$

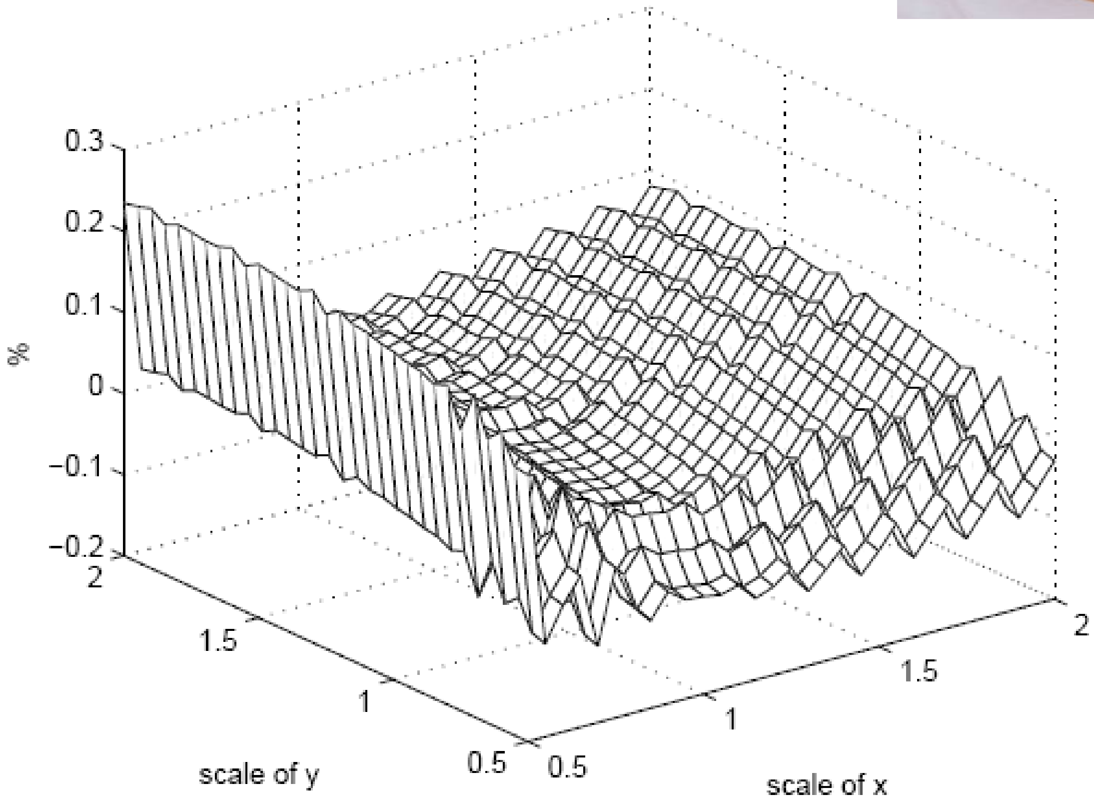
Aspect-ratio invariants

$$x' = a \cdot x$$
$$y' = b \cdot y$$

$$\mu'_{pq} = a^{p+1} b^{q+1} \mu_{pq}$$

$$A_{pq} = \frac{\mu_{00}^{(p+q+2)/2}}{\mu_{20}^{(p+1)/2} \cdot \mu_{02}^{(q+1)/2}} \cdot \mu_{pq}$$

Aspect-ratio invariants



Invariants to contrast changes

$$f'(x,y) = a \cdot f(x,y)$$

$$\mu'_{pq} = a \cdot \mu_{pq}$$

$$C_{pq} = \frac{\mu_{pq}}{\mu_{00}}$$

Invariants to contrast and TRS

$$\Phi'(p,q) = a^{p-q+1}\Phi(p,q)$$

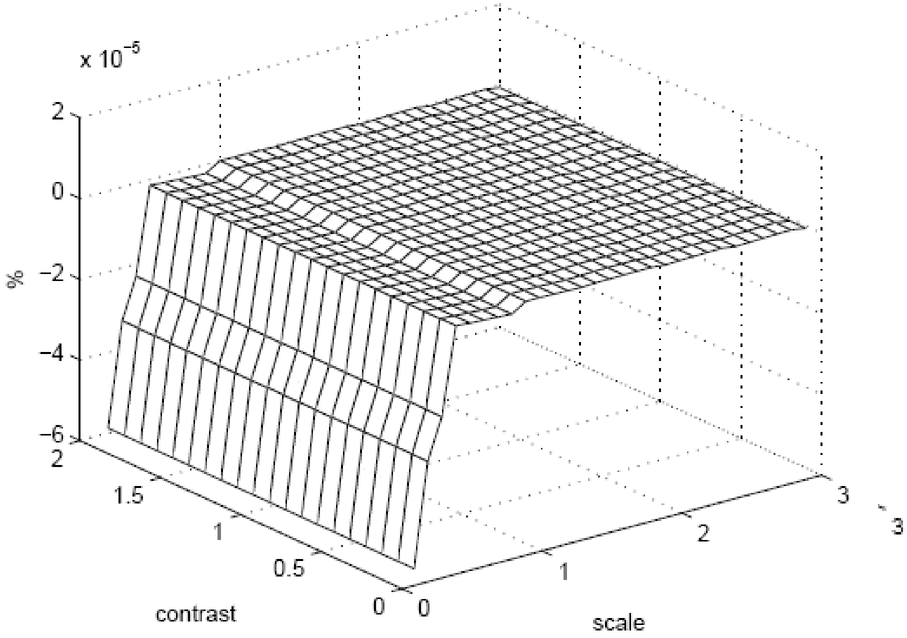
$$\widetilde{c_{pq}} = \frac{c_{pq}}{\mu_{00}^{\frac{p+q}{2}+1}}$$

$$\widetilde{c_{pq}}' = a^{-\frac{p+q}{2}}\widetilde{c_{pq}}$$

$$\Gamma(p,q) = \frac{\widetilde{\Phi(p,q)}}{|\widetilde{c_{pq}}| \cdot \widetilde{\Phi(p_0,q_0)}^{\frac{p-q}{2}}}$$

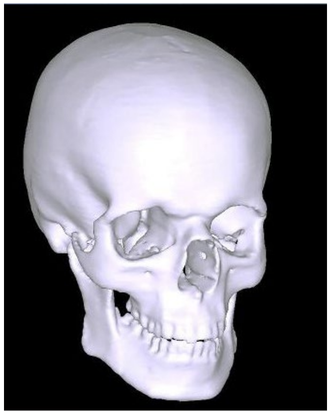
$$\widetilde{\Phi(p,q)}' = a^{-\frac{(p_0+q_0)(p-q)+(p+q)}{2}}\widetilde{\Phi(p,q)}$$

Invariants to contrast and TRS



TRS invariants in 3D

Important in medical imaging (MRI, CT, ...) and stereovision



3D geometric moments

$$m_{pqs} = \int\int\int x^p y^q z^s f(x, y, z) dx dy dz$$

TRS invariants in 3D

3D central moments

$$\mu_{pqs} = \int\int\int (x-x_t)^p (y-y_t)^q (z-z_t)^s f(x,y,z)\,dx\,dy\,dz$$

3D scale-normalized moments

$$\nu_{pqs} = \frac{\mu_{pqs}}{\mu_{000}^{w}} \qquad w = \frac{p+q+s}{3} + 1$$

TRS invariants in 3D

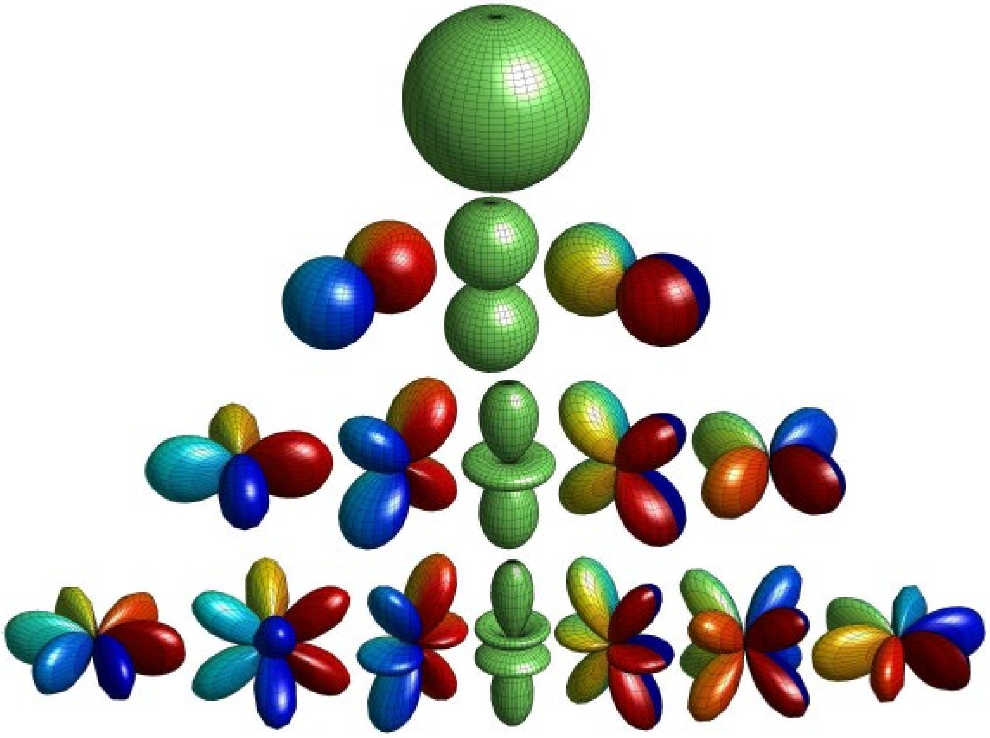
Simple 3D rotation invariants

$$\phi_1 = \mu_{200} + \mu_{020} + \mu_{002}$$

$$\begin{aligned}\phi_2 = \; & \mu_{020}\mu_{002} + \mu_{200}\mu_{002} + \mu_{200}\mu_{020} \\ & -(\mu_{011}^2 + \mu_{101}^2 + \mu_{110}^2)\end{aligned}$$

TRS invariants in 3D

Theory based on spherical harmonics



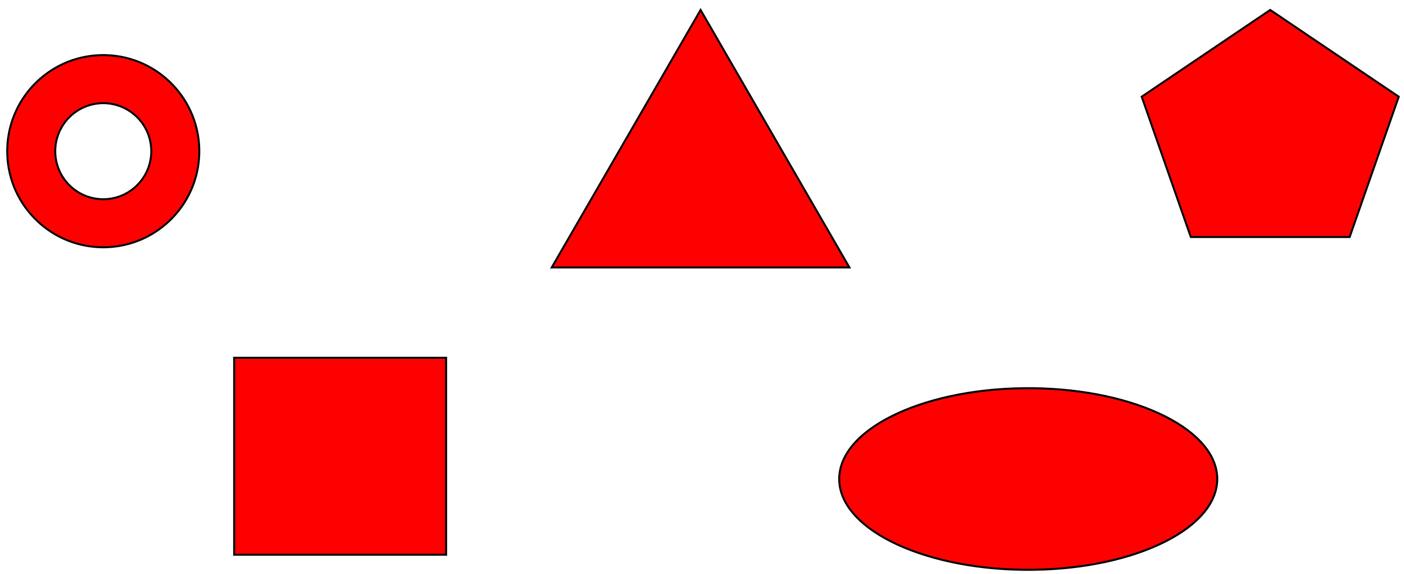
Documented applications of TRS moment invariants

- Recognition of aircraft and ship silhouettes
- Character/digit/symbol recognition
- Recognition of components on an assembly belt
- Image registration (medical, satellite, ...)
- Normalization of database images

Back to the rotation invariants in 2D

$$I = \prod_{i=1}^{n} c_{p_i q_i}^{k_i} \qquad \sum_{i=1}^{n} k_i (p_i - q_i) = 0$$

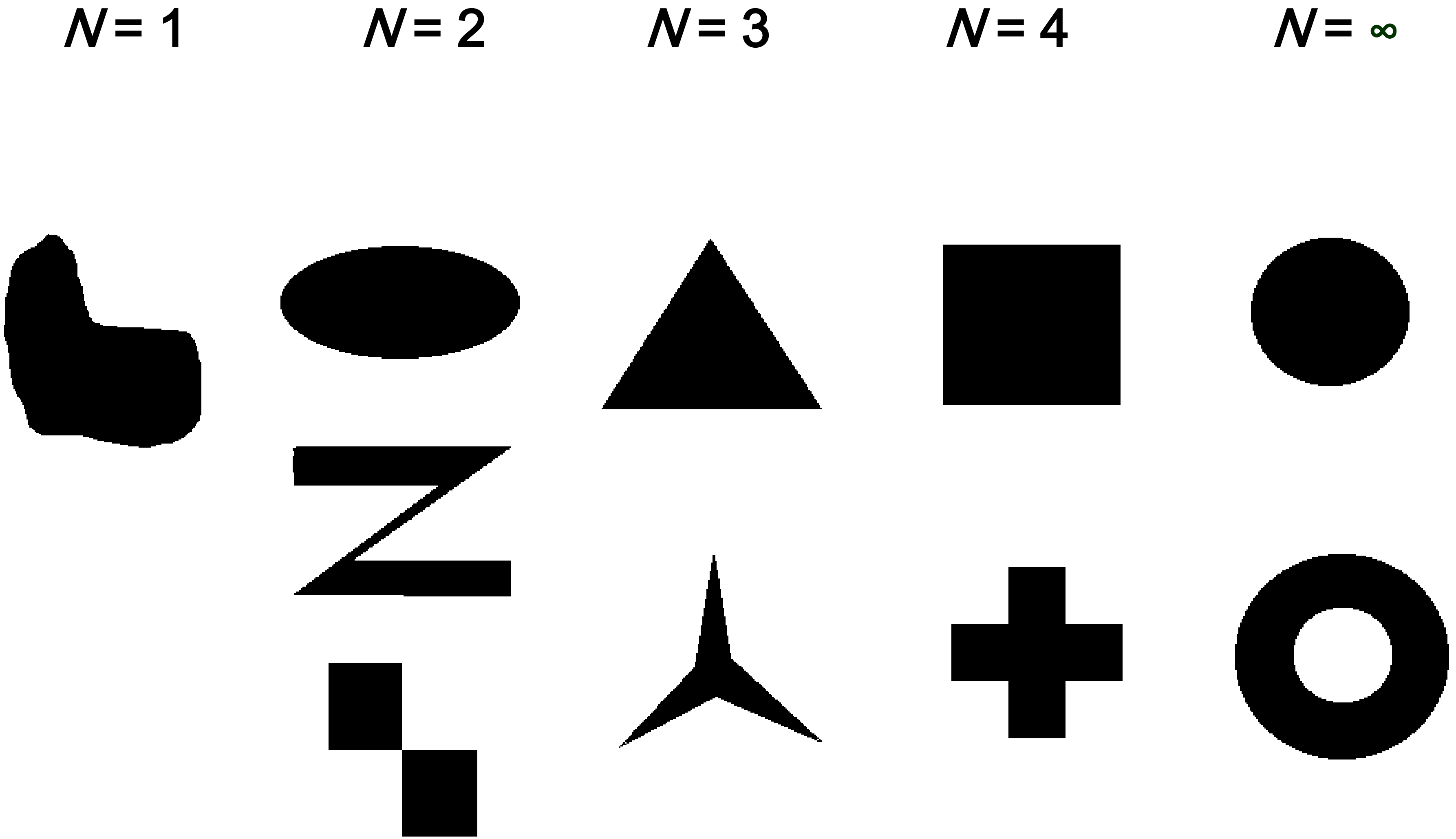
Recognizing symmetric objects



N-fold rotation symmetry

N-fold rotation symmetry: the object repeats itself when rotating by $2\pi j/N$ for all $j = 1, \cdots, N$.

Examples of N-fold RS



Qestion: *N*-fold symmetry versus axial symmetry

Axial symmetry S → N-fold symmetry,
S=N

N-fold symmetry → no axial symmetry
or S=N

Difficulties with symmetric objects

Many moments and many invariants are zero

If $f(x, y)$ has N–fold rotation symmetry then

$$c_{pq} = 0$$

for every p, q such that $(p - q)/N$ is not an integer.

Proof

$$c'_{pq} = e^{-i(p-q)\alpha} \cdot c_{pq}$$

$$\alpha = 2\pi / N$$

$$c'_{pq} = e^{\frac{-2\pi i(p-q)}{N}} \cdot c_{pq}$$

Proof (continued)

$$c'_{pq} = c_{pq} \quad \Rightarrow \quad c_{pq} = 0 \quad \vee \quad e^{\frac{-2\pi i(p-q)}{N}} = 1$$

If $(p-q)/N$ is not an integer then $e^{\frac{-2\pi i(p-q)}{N}} \neq 1$

$$\Rightarrow \quad c_{pq} = 0 \qquad \square$$

Difficulties with symmetric objects

The greater *N*, the less nontrivial invariants

Particularly

- $N = 1$ (no symmetry) $\Longrightarrow$ previous case
- $N = 2$ (central symmetry) $\Longrightarrow$ only even-order invariants exist
- $N = \infty$ (circular symmetry) $\Longrightarrow$ only $\Phi(p,p) \equiv c_{pp}$

Difficulties with symmetric objects

It is very important to use only non-trivial invariants

The choice of appropriate invariants (basis of invariants) depends on N

The basis for *N*-fold symmetric objects

Generalization of the theorem about the basis

$$\forall p, q : \qquad \Phi(p, q) \equiv c_{pq} c_{q_0 p_0}^{k}$$

$k = (p - q)/N$ is an integer

$$p + q \leq r$$

$$p \geq q$$

$$p_0 + q_0 \leq r$$

$$p_0 - q_0 = N$$

$$c_{p_0 q_0} \neq 0$$

$\langle \mathcal{B} \rangle$ – all rotation invariants generated from $\mathcal{B}$

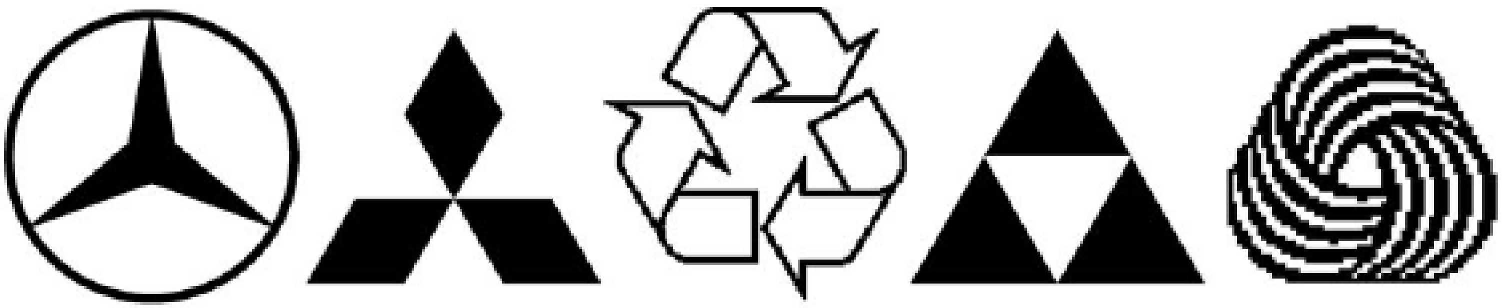
- M, N finite, L – least common multiple
$$\langle \mathcal{B}_M \rangle \cap \langle \mathcal{B}_N \rangle = \langle \mathcal{B}_L \rangle$$
If M/N is integer then $\langle \mathcal{B}_M \rangle \subset \langle \mathcal{B}_N \rangle$

-
$$\bigcap_{N=1}^{\infty} \langle \mathcal{B}_N \rangle = \langle \mathcal{B}_\infty \rangle$$

-
$$|\mathcal{B}_N| = \sum_{j=0}^{n} [\frac{r - jN + 2}{2}]$$
where $n = [r/N]$

$$|\mathcal{B}_\infty| = [\frac{r + 2}{2}]$$

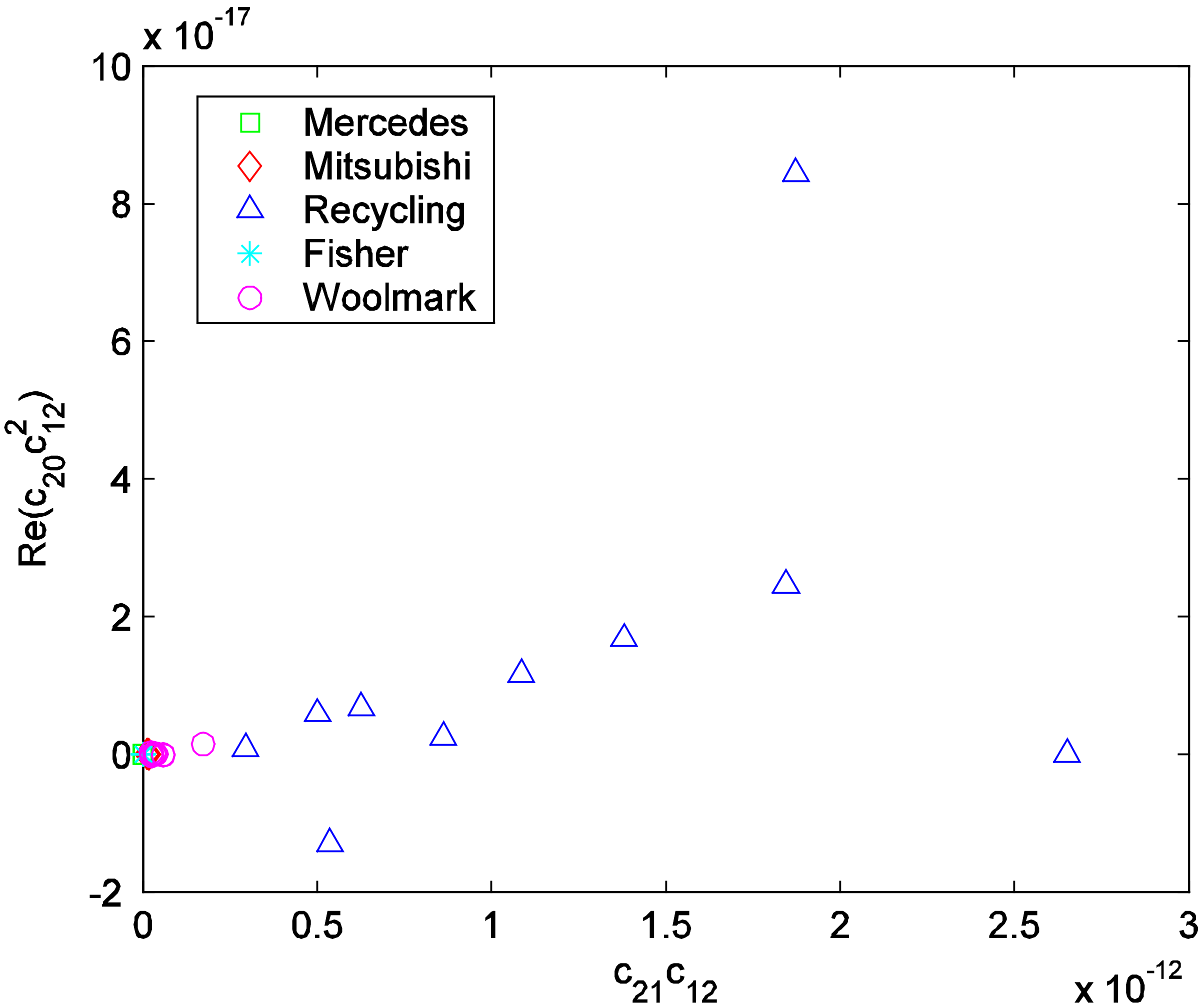
Recognition of symmetric objects – Experiment 1



5 objects with $N = 3$

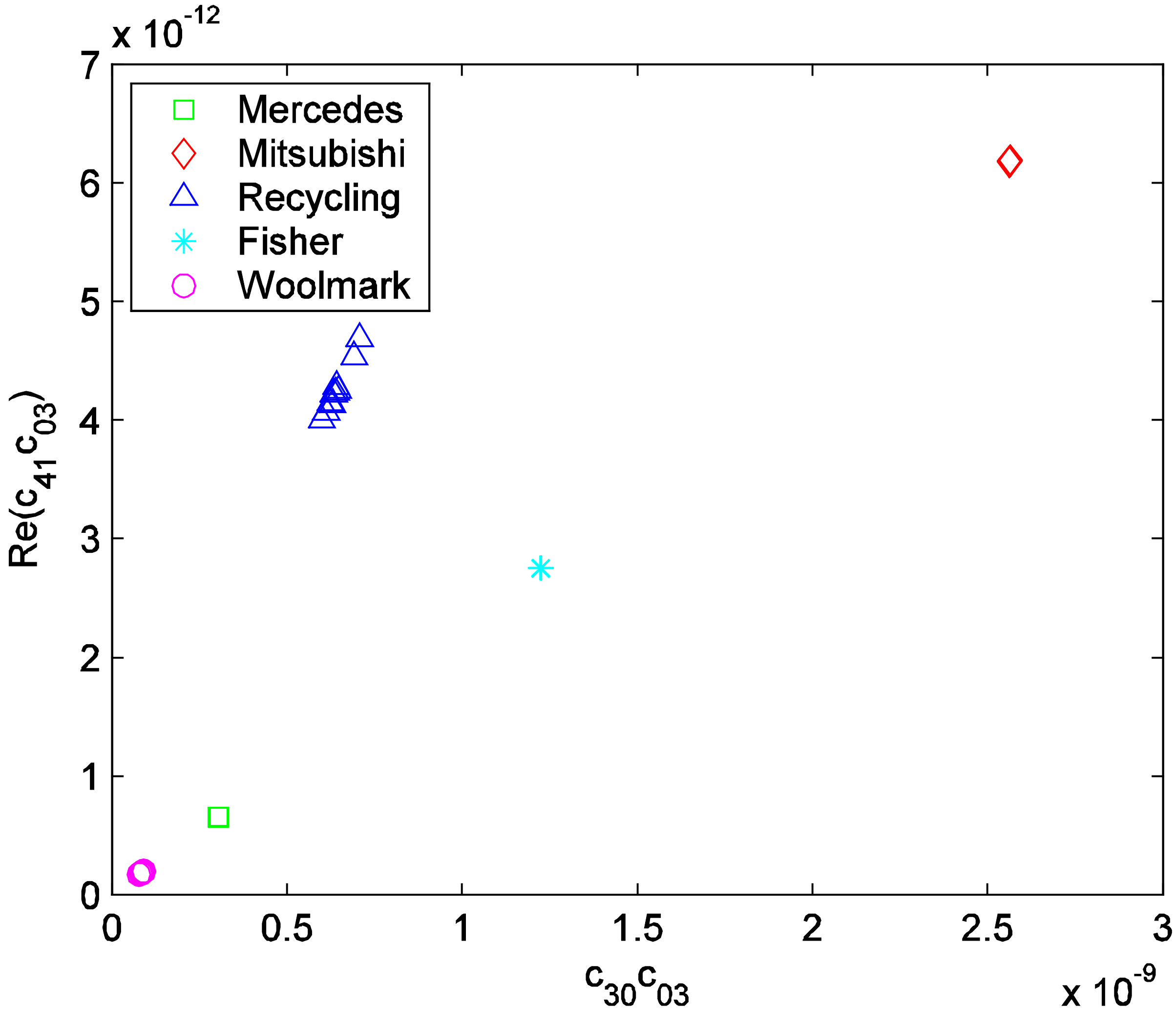
Recognition of symmetric objects – Experiment 1

Bad choice: $p_0 = 2$, $q_0 = 1$



Recognition of symmetric objects – Experiment 1

Optimal choice: $p_0 = 3$, $q_0 = 0$



Recognition of symmetric objects – Experiment 2

2 objects with $N = 1$

2 objects with $N = 2$

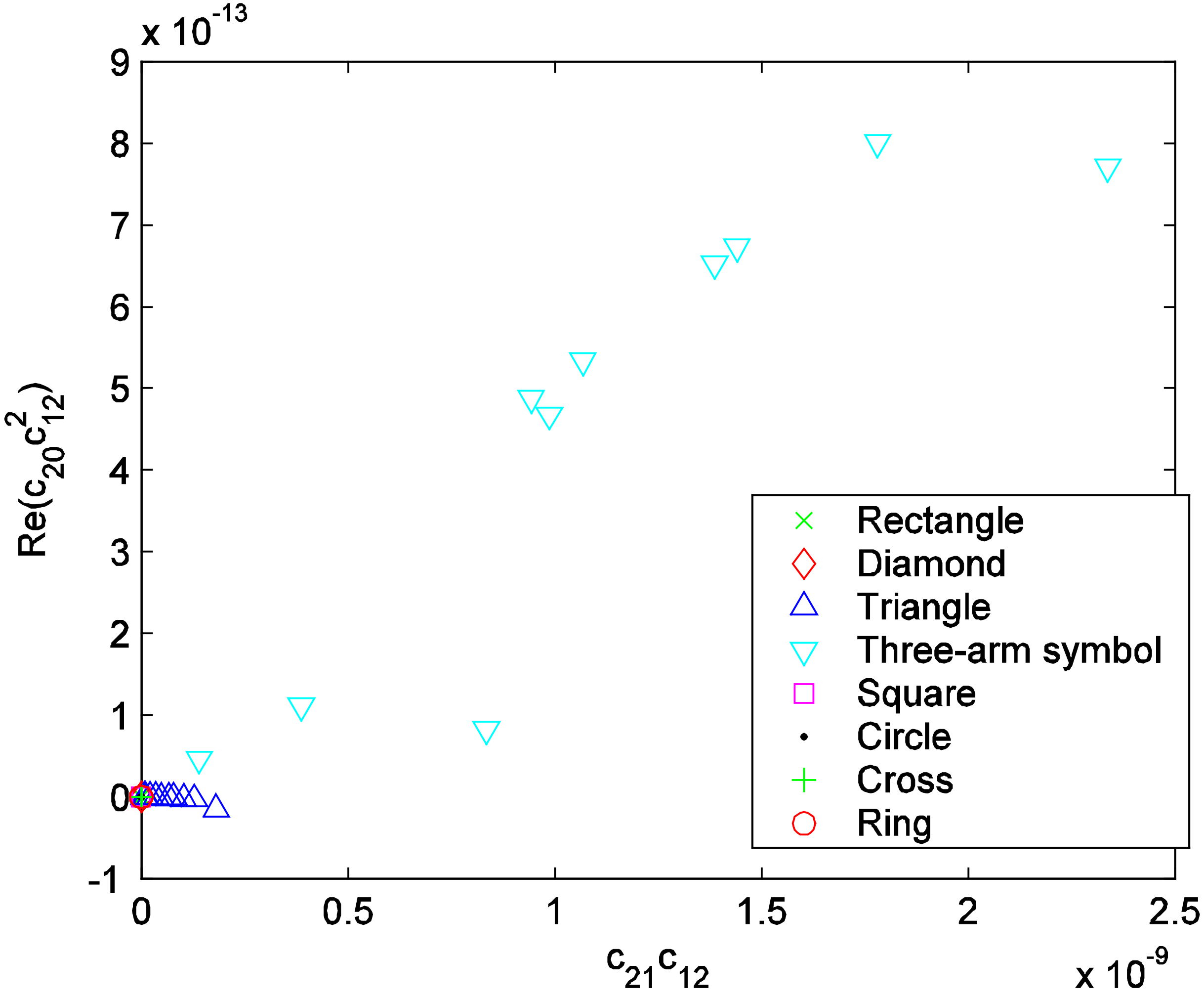
2 objects with $N = 3$

1 object with $N = 4$

2 objects with $N = \infty$

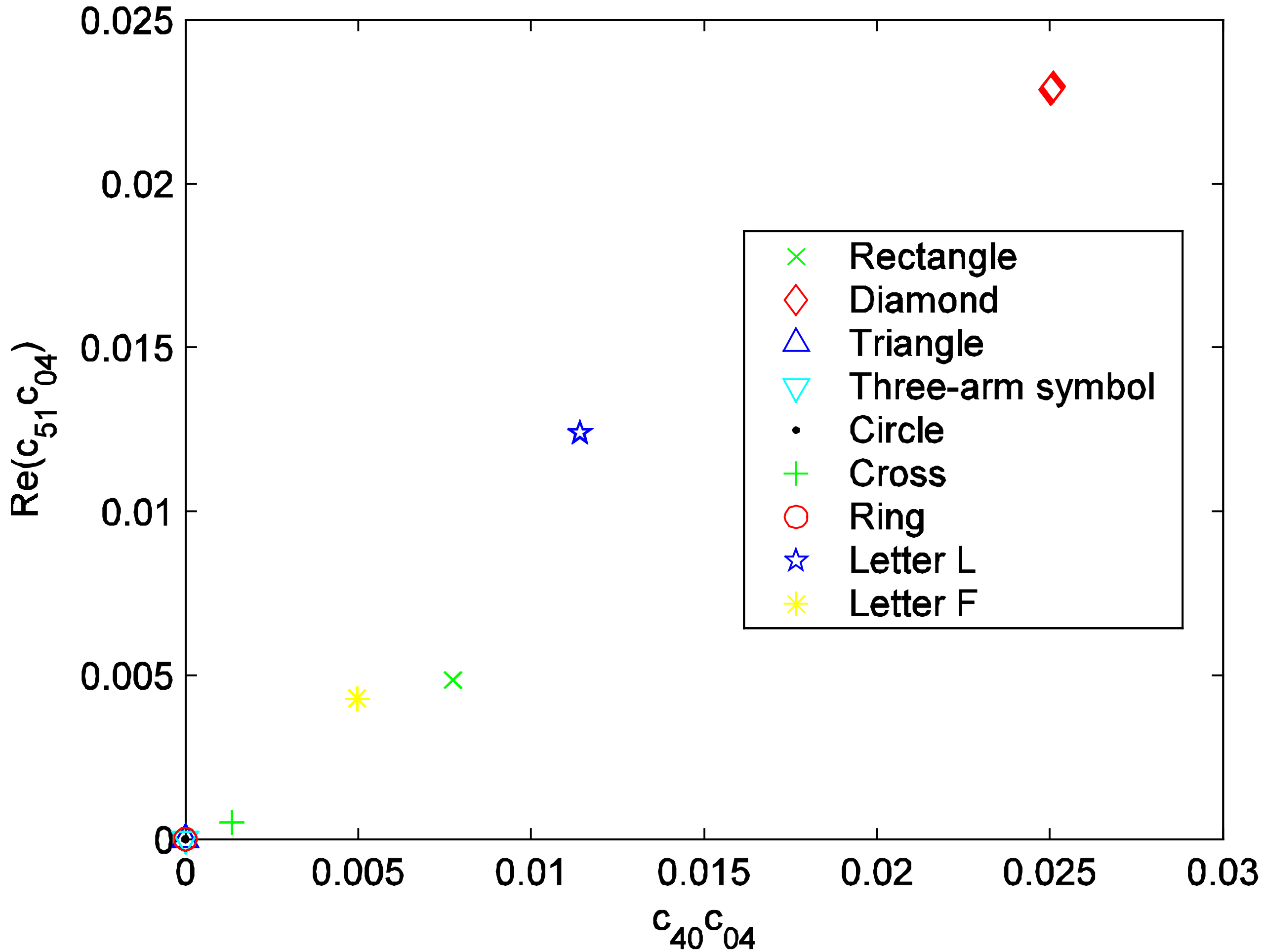
Recognition of symmetric objects – Experiment 2

Bad choice: $p_0 = 2$, $q_0 = 1$



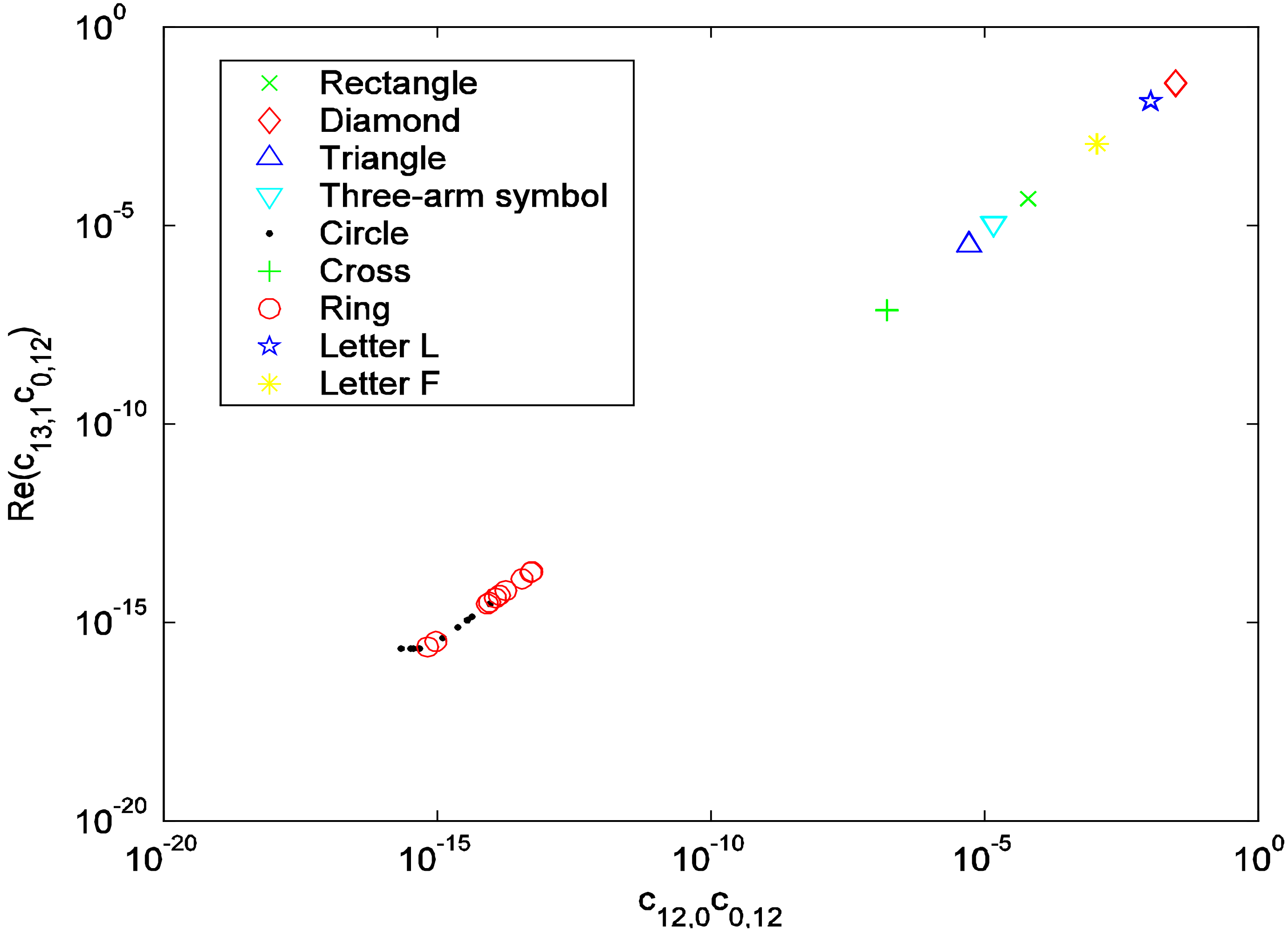
Recognition of symmetric objects – Experiment 2

Better but not optimal choice: $p_0 = 4$, $q_0 = 0$



Recognition of symmetric objects – Experiment 2

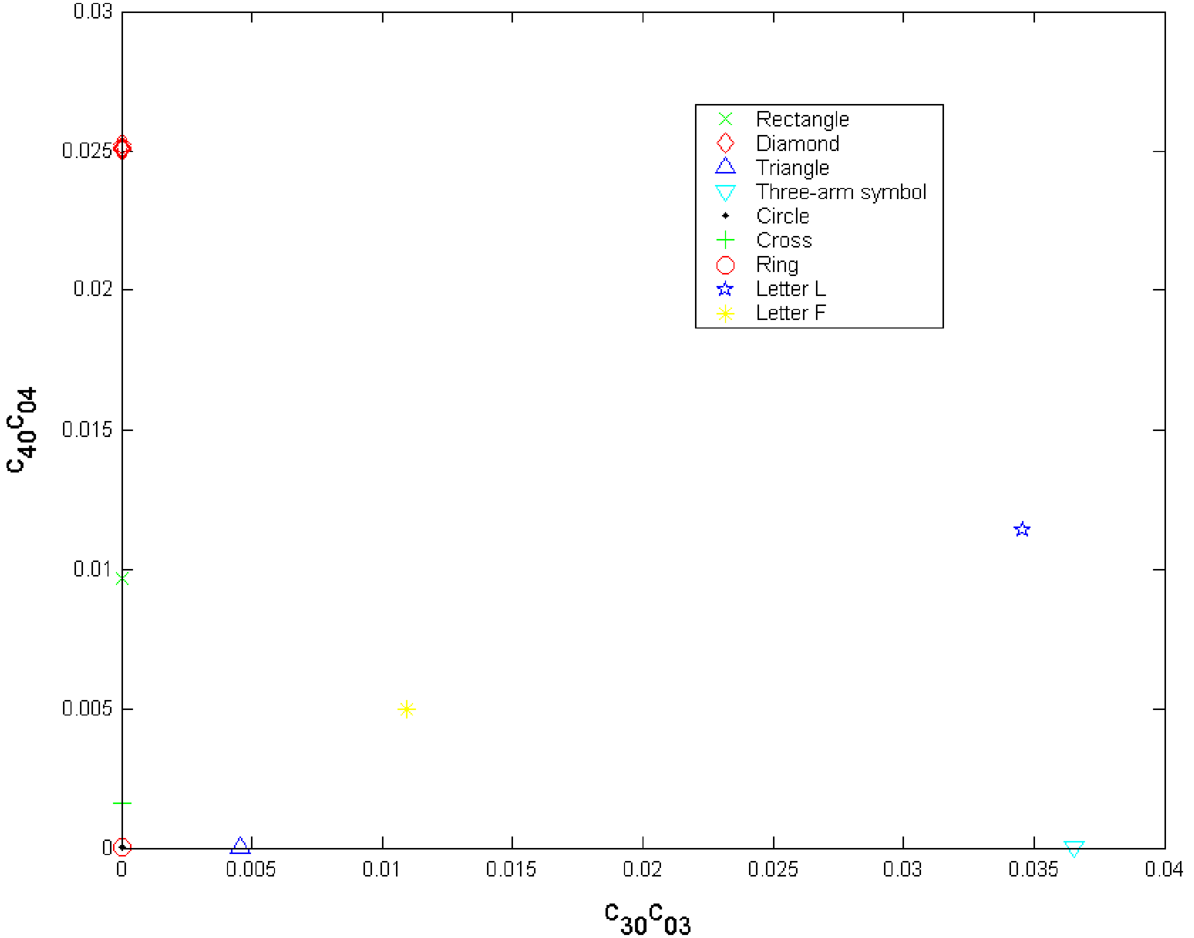
Theoretically optimal choice: $p_0 = 12$, $q_0 = 0$



Logarithmic scale

Recognition of symmetric objects – Experiment 2

The best choice: mixed orders



Normalization of symmetric objects by complex moments

Constraint

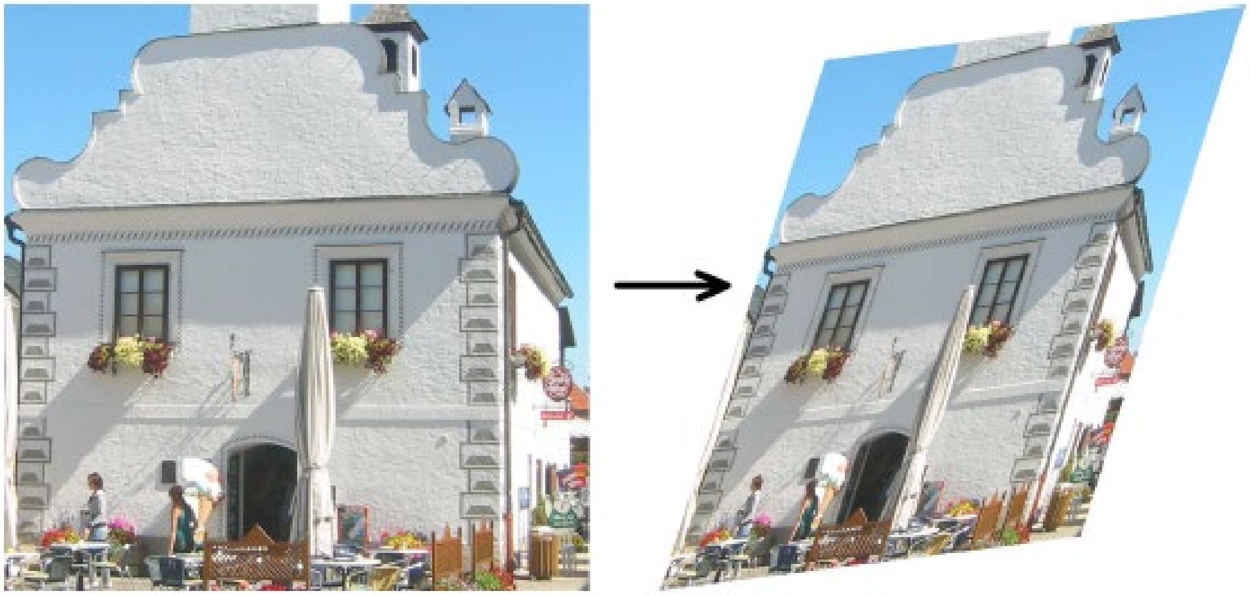
$$c_{st} \text{ real and positive} \Longrightarrow$$
$$\alpha = \frac{1}{s-t} \cdot \arctan\left(\frac{\mathrm{Im} c_{st}}{\mathrm{Re} c_{st}}\right)$$

c_{st} must be non-zero !

Invariants to affine transform

What is affine transform?

$$u = a_0 + a_1 x + a_2 y$$
$$v = b_0 + b_1 x + b_2 y$$



Why is affine transform important?

- Affine transform is a good approximation of projective transform

$$u = \frac{a_0 + a_1 x + a_2 y}{1 + c_1 x + c_2 y}$$

$$v = \frac{b_0 + b_1 x + b_2 y}{1 + c_1 x + c_2 y}$$

- Projective transform describes a perspective projection of 3-D objects onto 2-D plane by a central camera

Why not projective moment invariants?

- Do not exist when using any finite set of moments
- Do not exist when using infinite set of (all) moments
- Exist formally as infinite series of moments of both positive and **negative** indexes

Affine moment invariants

Many ways how to derive them

- Theory of algebraic invariants (Fundamental theorem)
- Graph method
- Image normalization
- Cayley-Aronhold equation
- Hybrid approaches

All methods lead to equivalent invariants …

… such as

$$I_1 = (\mu_{20}\mu_{02} - \mu_{11}^2)/\mu_{00}^4$$

$$I_2 = (\mu_{30}^2\mu_{03}^2 - 6\mu_{30}\mu_{21}\mu_{12}\mu_{03} + 4\mu_{30}\mu_{12}^3$$
$$+4\mu_{03}\mu_{21}^3 - 3\mu_{21}^2\mu_{12}^2)/\mu_{00}^{10}$$

Two simplest AMI's, frequently cited

AMI's by means of the Fundamental theorem

Binary algebraic form

$$\sum_{k=0}^{p} \binom{p}{k} a_k x^{p-k} y^k$$

Algebraic invariant of weight *w*

$$I(a_0', a_1', \ldots, a_{p_a}'; b_0', b_1', \ldots, b_{p_b}'; \ldots) = J^w I(a_0, a_1, \ldots, a_{p_a}; b_0, b_1, \ldots, b_{p_b}; \ldots)$$

AMI's by means of the Fundamental theorem

Theorem 3.1 *(Fundamental theorem of AMIs) If the binary forms of orders p_a, p_b, … have an algebraic invariant of weight w and degree r*

$$I(a'_0, a'_1, \ldots, a'_{p_a}; b'_0, b'_1, \ldots, b'_{p_b}; \ldots) = J^w I(a_0, a_1, \ldots, a_{p_a}; b_0, b_1, \ldots, b_{p_b}; \ldots) \ ,$$

then the moments of the same orders have the same invariant but with the additional factor $|J|^r$:

$$I(\mu'_{p_a 0}, \mu'_{p_a -1,1}, \ldots, \mu'_{0 p_a}; \mu'_{p_b 0}, \mu'_{p_b -1,1}, \ldots, \mu'_{0 p_b}; \ldots) =$$

$$= J^w |J|^r I(\mu_{p_a 0}, \mu_{p_a -1,1}, \ldots, \mu_{0 p_a}; \mu_{p_b 0}, \mu_{p_b -1,1}, \ldots, \mu_{0 p_b}; \ldots).$$

AMI's by means of the graph method

$(x_1, y_1), (x_2, y_2)$ - arbitrary points

$$C_{12} = x_1 y_2 - x_2 y_1$$

$$C'_{12} = J \cdot C_{12}$$

r points, n_{kj} – non-negative integers

$$I(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{k,j=1}^{r} C_{kj}^{n_{kj}} \cdot \prod_{i=1}^{r} f(x_i, y_i)\ \mathrm{d}x_i\ \mathrm{d}y_i$$

AMI's by means of the graph method

$$I(f)' = J^{w}|J|^{r} \cdot I(f)$$

where

$$w = \sum_{k,j} n_{kj}$$

Affine Moment Invariants

$$\left(\frac{I(f)}{\mu_{00}^{w+r}}\right)' = (\text{sign } J)^w \left(\frac{I(f)}{\mu_{00}^{w+r}}\right)$$

if *w* is even → "true" invariants

if *w* is odd → "pseudoinvariants"

Simple examples of the AMI's

1) $r = 2,\ n_{12} = 2$

$$I(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 y_2 - x_2 y_1)^2 f(x_1, y_1) f(x_2, y_2)\ \mathrm{d}x_1\ \mathrm{d}y_1\ \mathrm{d}x_2\ \mathrm{d}y_2$$

$$= 2(m_{20} m_{02} - m_{11}^2)$$

$$I_1 = (\mu_{20}\mu_{02} - \mu_{11}^2)/\mu_{00}^4$$

Simple examples of the AMI's

2) $r=3,\ n_{12}=2, n_{13}=2, n_{23}=0$

$$I(f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1y_2-x_2y_1)^2(x_1y_3-x_3y_1)^2$$

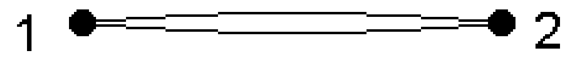
$$f(x_1,y_1)f(x_2,y_2)f(x_3,y_3)\ \mathrm{d}x_1\ \mathrm{d}y_1\ \mathrm{d}x_2\ \mathrm{d}y_2\ \mathrm{d}x_3\ \mathrm{d}y_3$$

$$=m_{20}^2m_{04}-4m_{20}m_{11}m_{13}+2m_{20}m_{02}m_{22}+4m_{11}^2m_{22}$$

$$-4m_{11}m_{02}m_{31}+m_{02}^2m_{40}$$

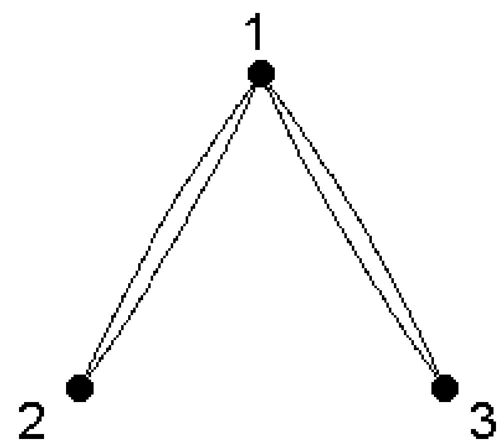
Graph representation of the AMI's

(x_k, y_k) – a node of the graph



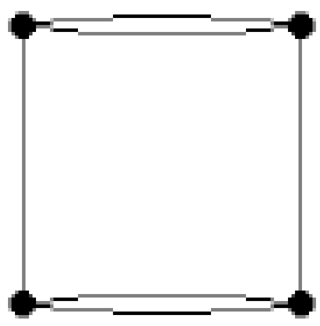
C_{kj} – an edge of the graph

$C_{kj}^{n_{kj}}$ – n_{kj} edges



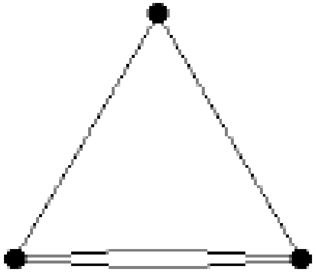
Graph representation of the AMI's

$$I_2 = (-\mu_{30}^2\mu_{03}^2 + 6\mu_{30}\mu_{21}\mu_{12}\mu_{03} - 4\mu_{30}\mu_{12}^3 - 4\mu_{21}^3\mu_{03} + 3\mu_{21}^2\mu_{12}^2)/\mu_{00}^{10}$$



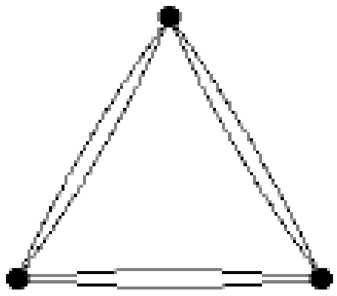
Graph representation of the AMI's

$$I_3 = (\mu_{20}\mu_{21}\mu_{03} - \mu_{20}\mu_{12}^2 - \mu_{11}\mu_{30}\mu_{03} + \mu_{11}\mu_{21}\mu_{12} + \mu_{02}\mu_{30}\mu_{12} - \mu_{02}\mu_{21}^2)/\mu_{00}^7$$



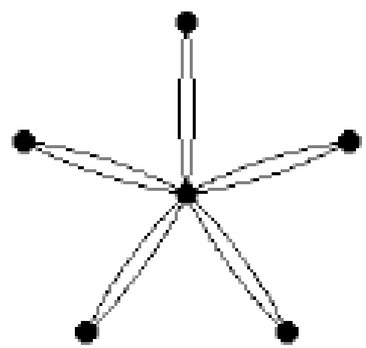
Graph representation of the AMI's

$$I_7 = (\mu_{40}\mu_{22}\mu_{04} - \mu_{40}\mu_{13}^2 - \mu_{31}^2\mu_{04} + 2\mu_{31}\mu_{22}\mu_{13} - \mu_{22}^3)/\mu_{00}^9$$



Graph representation of the AMI's

$$\begin{aligned}
I_{352} = {} & (\mu_{20}^5\mu_{0,10} - 10\mu_{20}^4\mu_{11}\mu_{19} + 5\mu_{20}^4\mu_{02}\mu_{28} + 40\mu_{20}^3\mu_{11}^2\mu_{28} - 40\mu_{20}^3\mu_{11}\mu_{02}\mu_{37} \\
& +10\mu_{20}^3\mu_{02}^2\mu_{46} - 80\mu_{20}^2\mu_{11}^3\mu_{37} + 120\mu_{20}^2\mu_{11}^2\mu_{02}\mu_{46} - 60\mu_{20}^2\mu_{11}\mu_{02}^2\mu_{55} \\
& +10\mu_{20}^2\mu_{02}^3\mu_{64} + 80\mu_{20}\mu_{11}^4\mu_{46} - 160\mu_{20}\mu_{11}^3\mu_{02}\mu_{55} + 120\mu_{20}\mu_{11}^2\mu_{02}^2\mu_{64} \\
& -40\mu_{20}\mu_{11}\mu_{02}^3\mu_{73} + 5\mu_{20}\mu_{02}^4\mu_{82} - 32\mu_{11}^5\mu_{55} + 80\mu_{11}^4\mu_{02}\mu_{64} - 80\mu_{11}^3\mu_{02}^2\mu_{73} \\
& +40\mu_{11}^2\mu_{02}^3\mu_{82} - 10\mu_{11}\mu_{02}^4\mu_{91} + \mu_{02}^5\mu_{10,0})/\mu_{00}^{16}
\end{aligned}$$



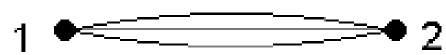
Dependence among invariants

- Trivial invariants (zero, identity)
- Reducible invariants (products, linear combinations)
- Irreducible invariants (polynomials, polynomials of products)
- Independent invariants

Dependence among invariants

- Trivial invariants (always zero or identical)

$$
\begin{aligned}
I(f) &= \int_{-\infty}^{\infty} (x_1 y_2 - x_2 y_1)^3 f(x_1, y_1) f(x_2, y_2) \, \mathrm{d}x_1 \, \mathrm{d}y_1 \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\
&= m_{30} m_{03} - 3 m_{21} m_{21} + 3 m_{21} m_{21} - m_{30} m_{03} = 0.
\end{aligned}
$$



Removing dependence

For w ≤ 12 :

2 533 942 752   invariants (graphs) altogether

2 532 349 394   zero invariants

1 575 126   identical invariants

14 538   linear combinations

2 105   products

1589 irreducible invariants

80 independent invariants

Removing dependence

The most difficult step: How to proceed from irreducible to independent invariants?

Exhaustive search of all possible polynomial dependencies

$$-4I_1^3I_2^2 + 12I_1^2I_2I_3^2 - 12I_1I_3^4 - I_2I_4^2 + 4I_3^3I_4 - I_5^2 = 0$$

$$-16I_1^3I_7^2 - 8I_1^2I_6I_7I_8 - I_1I_6^2I_8^2 + 4I_1I_6I_9^2 + 12I_1I_7I_8I_9 + I_6I_8^2I_9 - I_7I_8^3 - 4I_9^3 - I_{10}^2 = 0$$

The dependencies themselves may be dependent ! (2^{nd}-order dependencies)

Higher-order dependencies

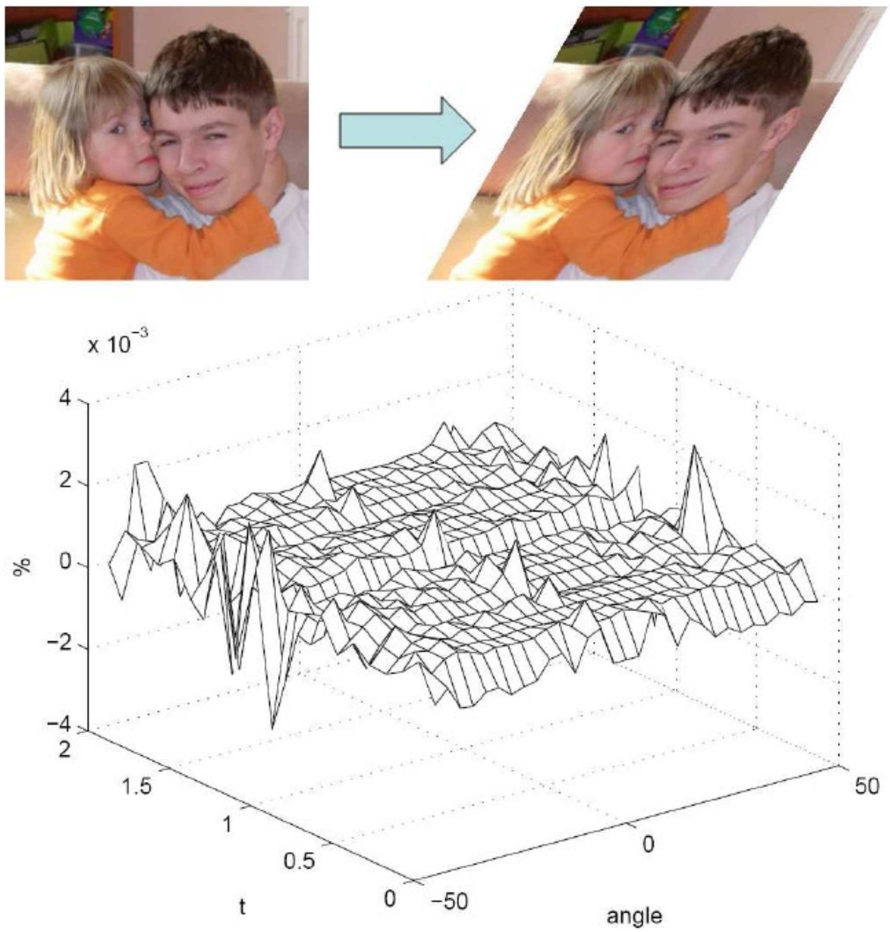
$$\begin{aligned} S_1: &\quad I_a^2 + I_b I_c &=& \quad 0, \\ S_2: &\quad I_d^2 - I_b I_c^2 &=& \quad 0, \\ S_3: &\quad I_a^4 + 2I_a^2 I_b I_c + I_d^2 I_b &=& \quad 0. \end{aligned}$$

$$S_1^2 + I_b S_2 - S_3 = 0$$

The number of independent invariants:

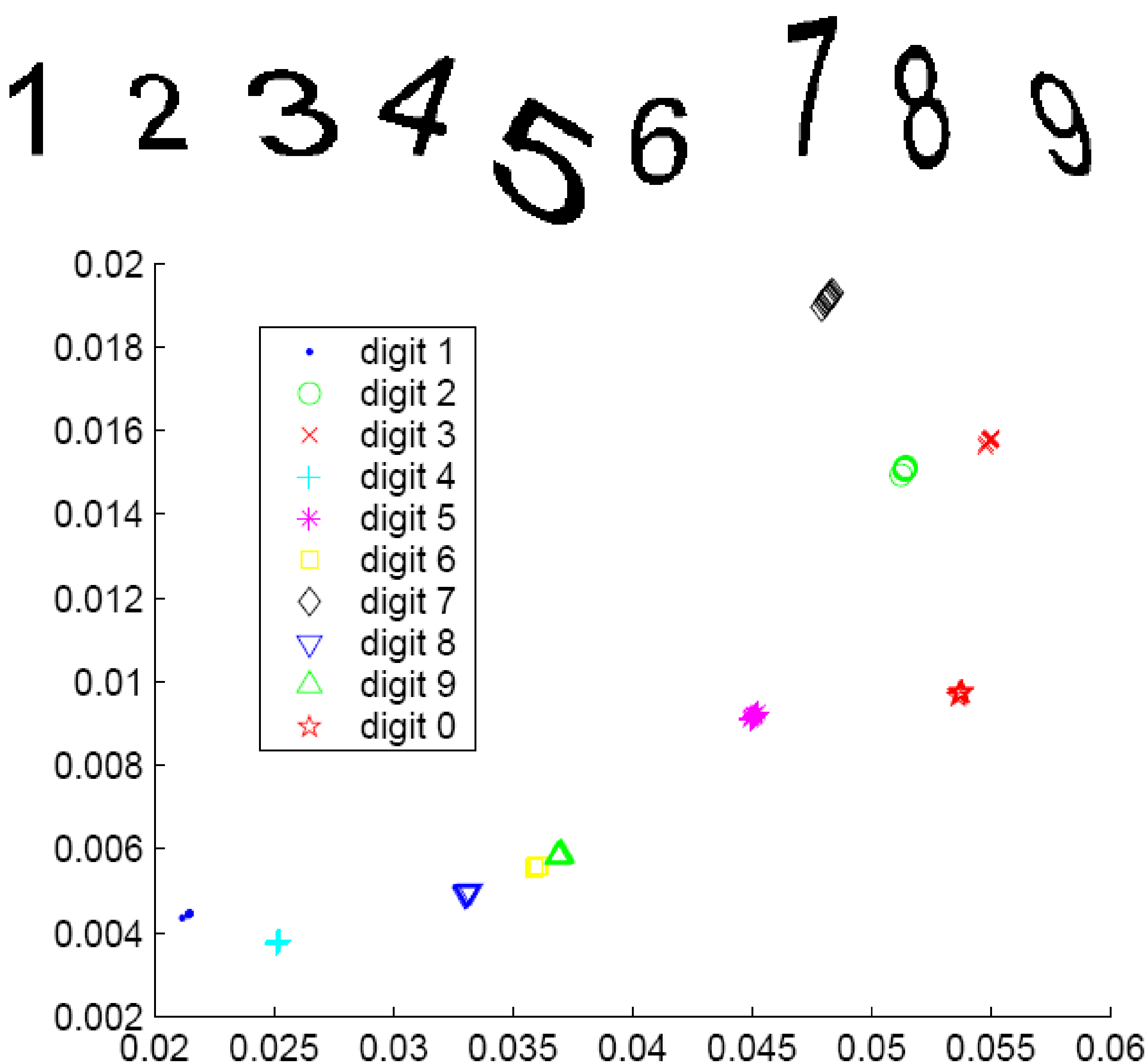
$$n = n_0 - n_1 + n_2 - n_3 + \ldots \quad ,$$

Numerical experiments with the AMI's



$$I_1 = (\mu_{20}\mu_{02} - \mu_{11}^2)/\mu_{00}^4$$

Digit recognition by the AMI's



Noisy digit recognition by the AMI's

a) Independent invariant set $I_1, I_2, I_3, I_4, I_6, I_7, I_8, I_9$

1	2	3	4	5	6	7	8	9	0	overall
76	80	31	34	100	46	76	100	68	51	66.2

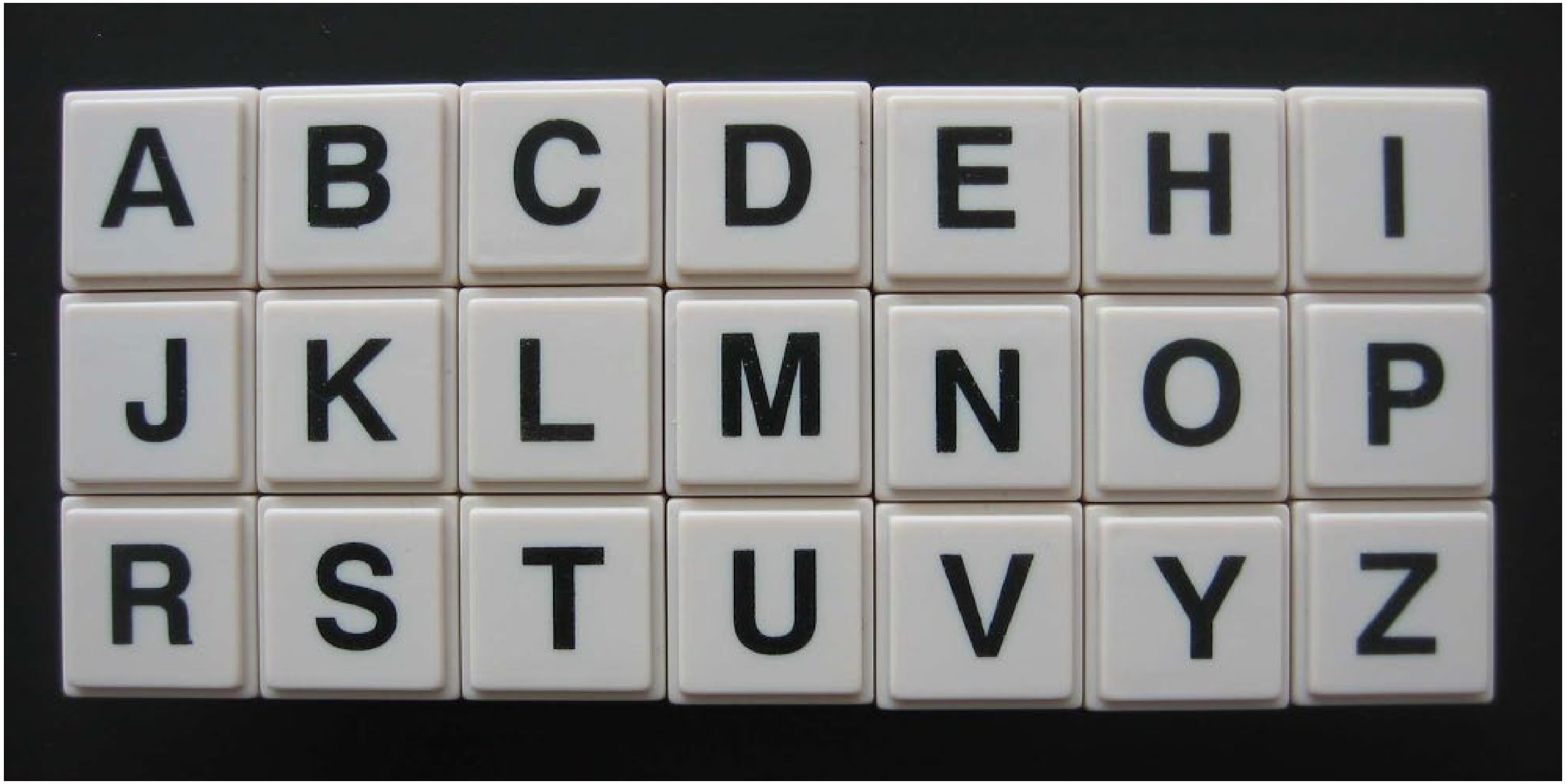
b) Dependent invariant set $I_1, I_2, I_3, I_4, I_6, I_7, I_8, I_9, I_{10}$

1	2	3	4	5	6	7	8	9	0	overall
51	100	46	73	53	56	74	100	72	70	69.5

c) Independent invariant set $I_1, I_2, I_3, I_4, I_6, I_7, I_8, I_9, I_{25}$

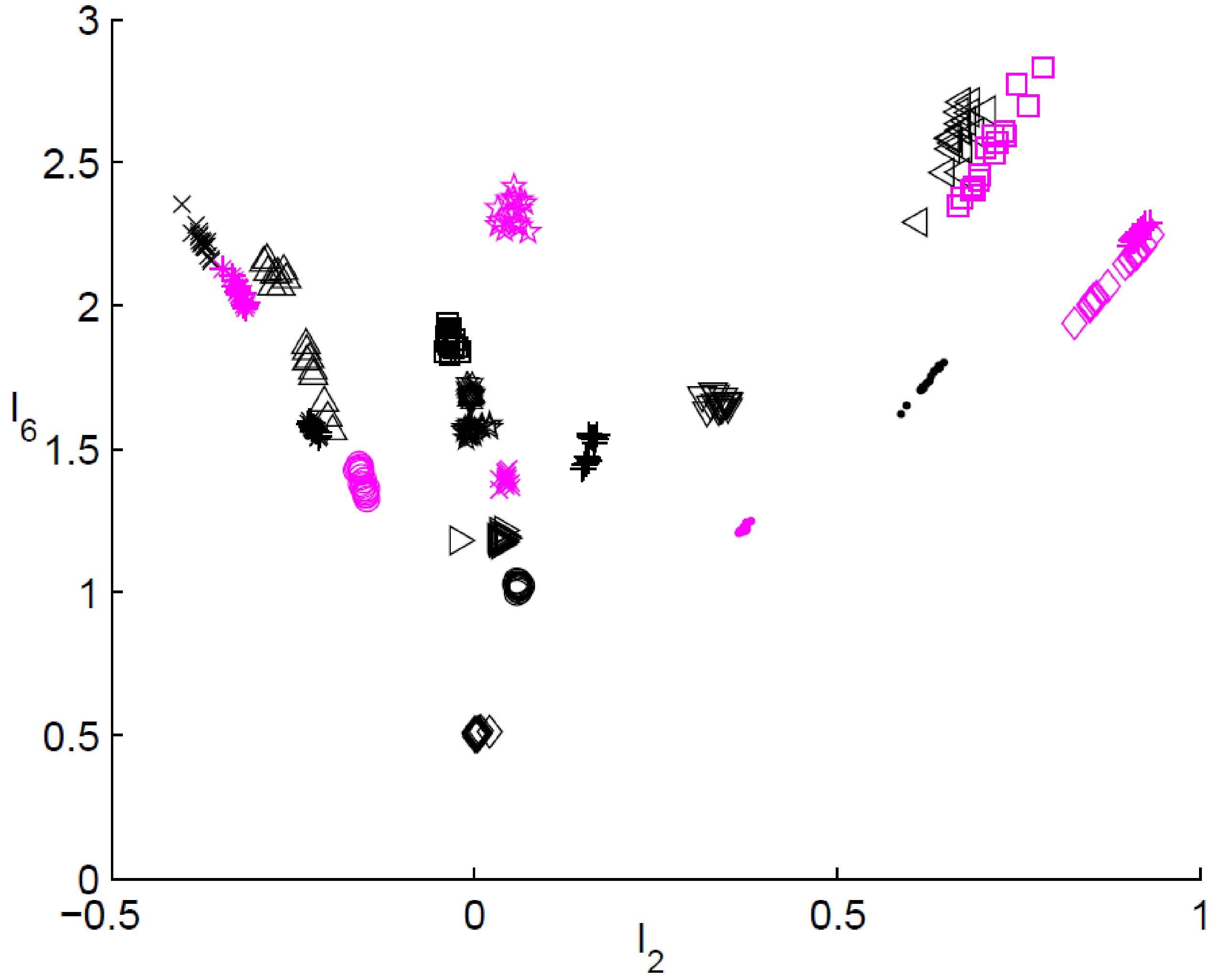
1	2	3	4	5	6	7	8	9	0	overall
89	77	73	63	94	66	79	100	54	50	74.5

Scrabble tiles recognition by the AMI's



Scrabble tiles recognition by the AMI's

Scrabble tiles recognition by the AMI's

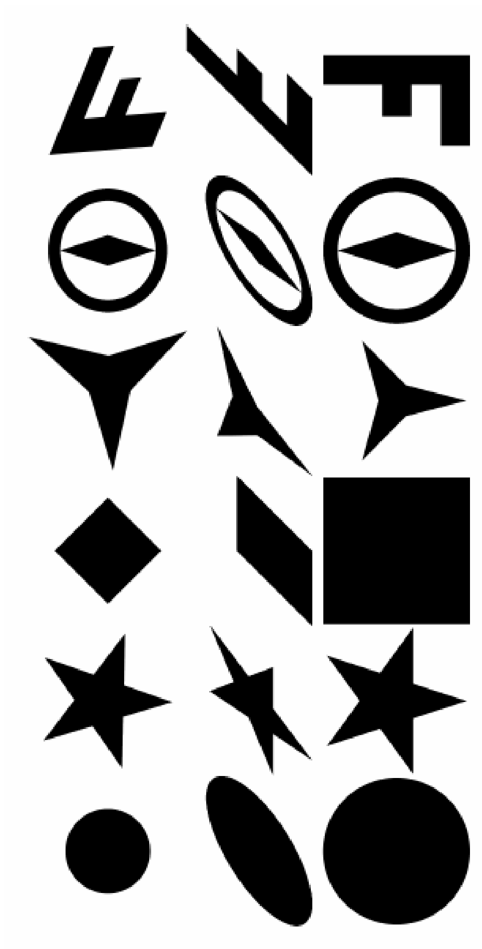


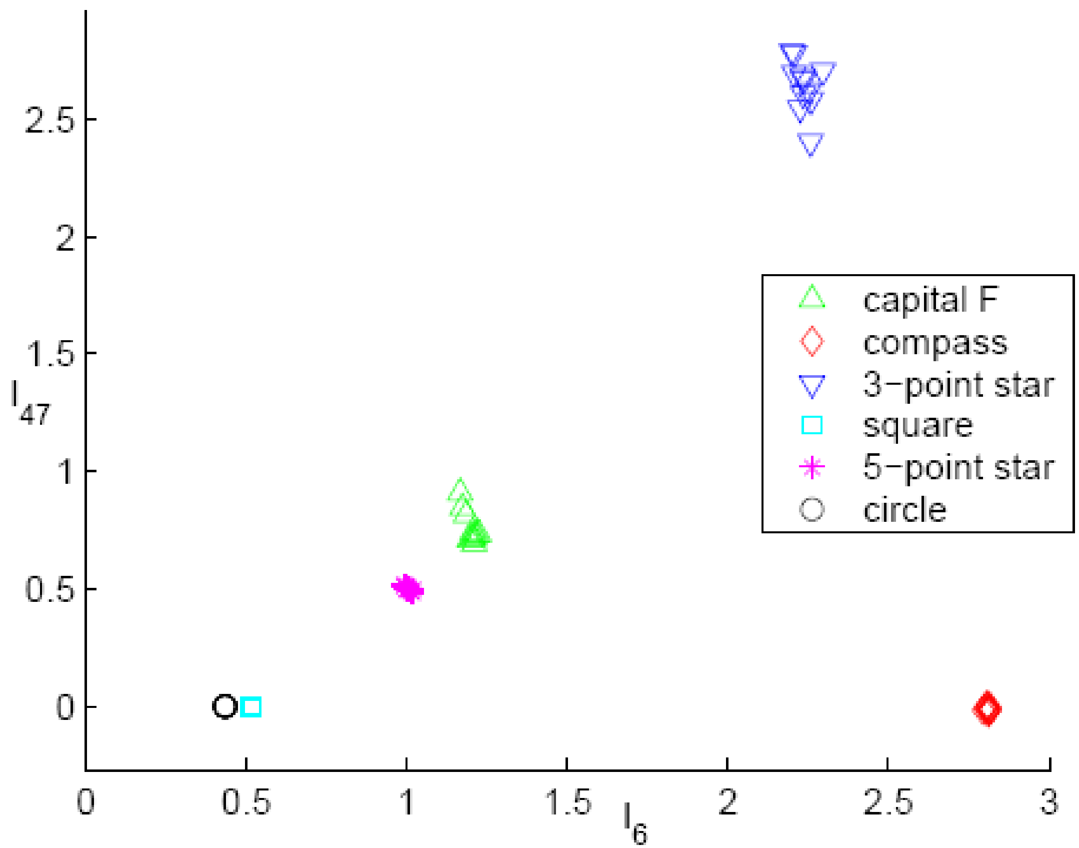
Scrabble tiles recognition – the results

- Dependent set of 17 AMI's → 9 misclassifications (out of 168 cases)

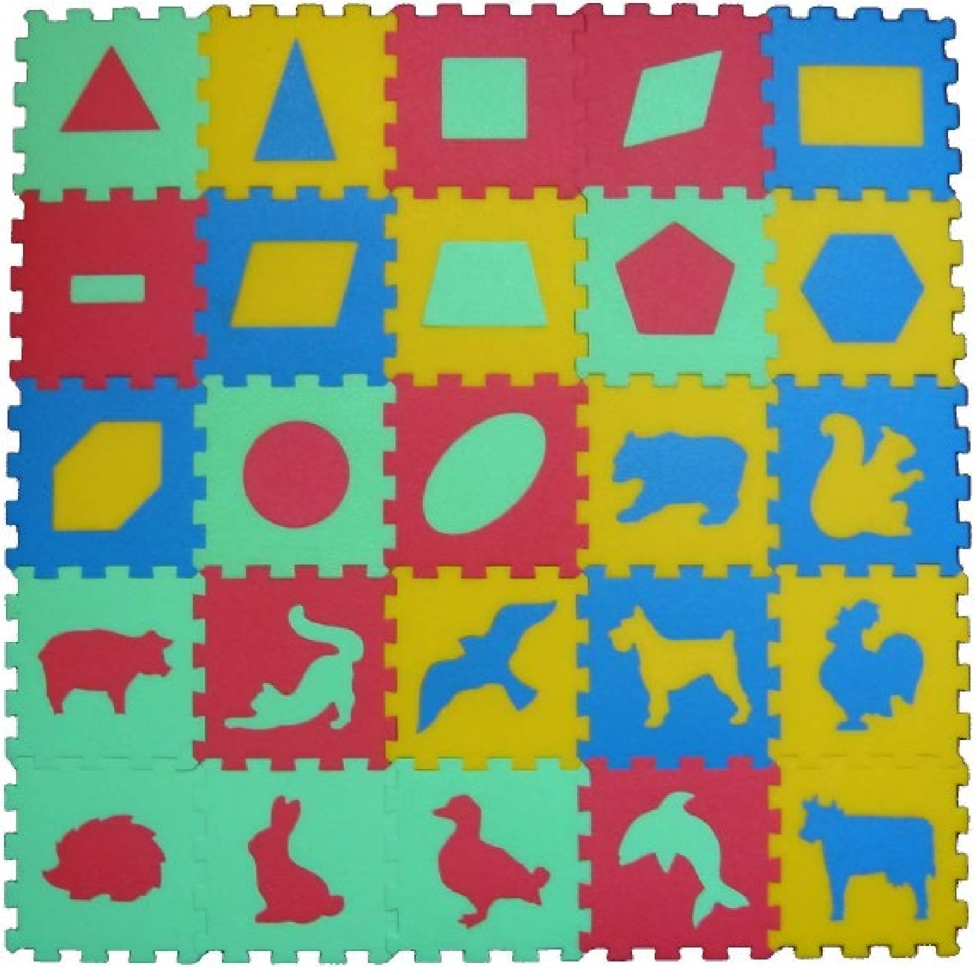
- Independent set of 15 AMI's → 5 misclassifications

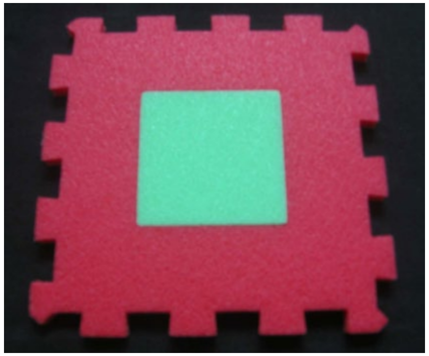
Recognition of symmetric patterns

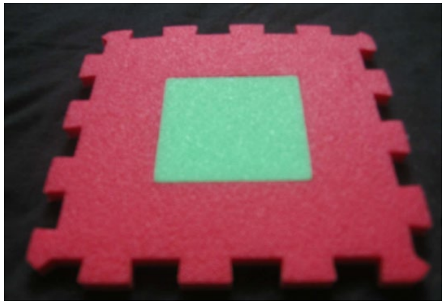


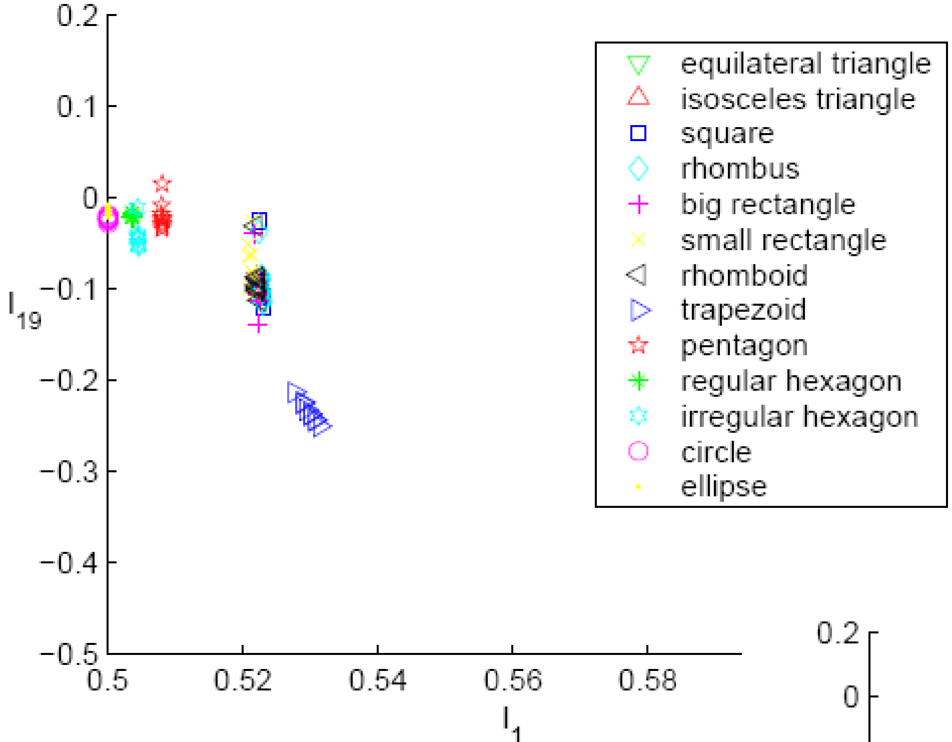


Recognition of children's mosaic

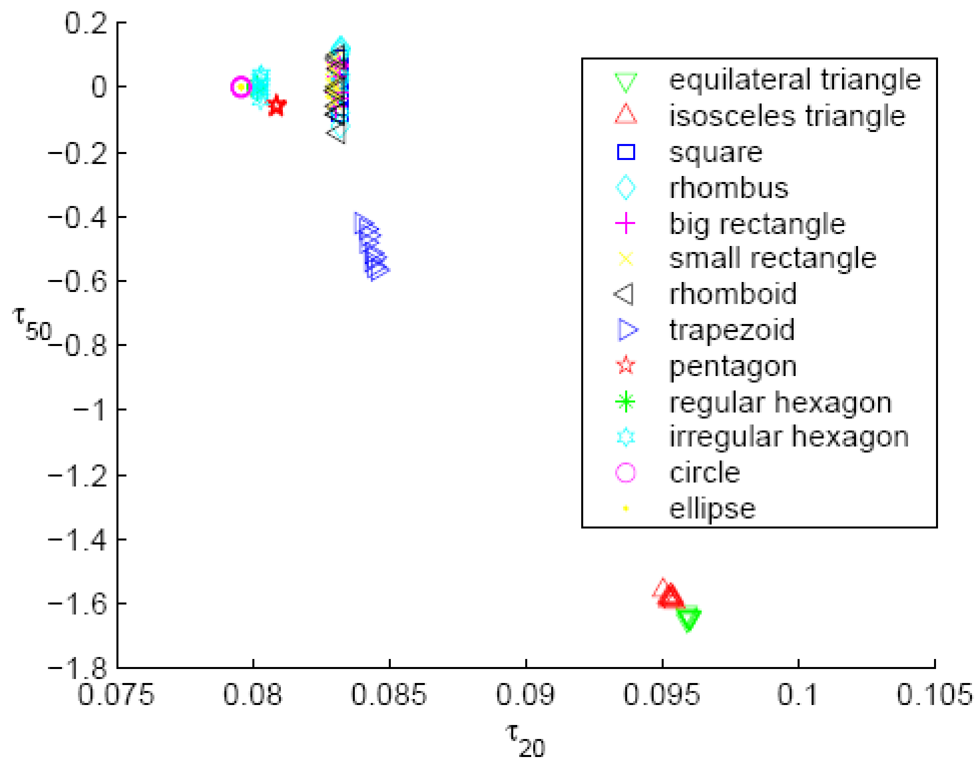




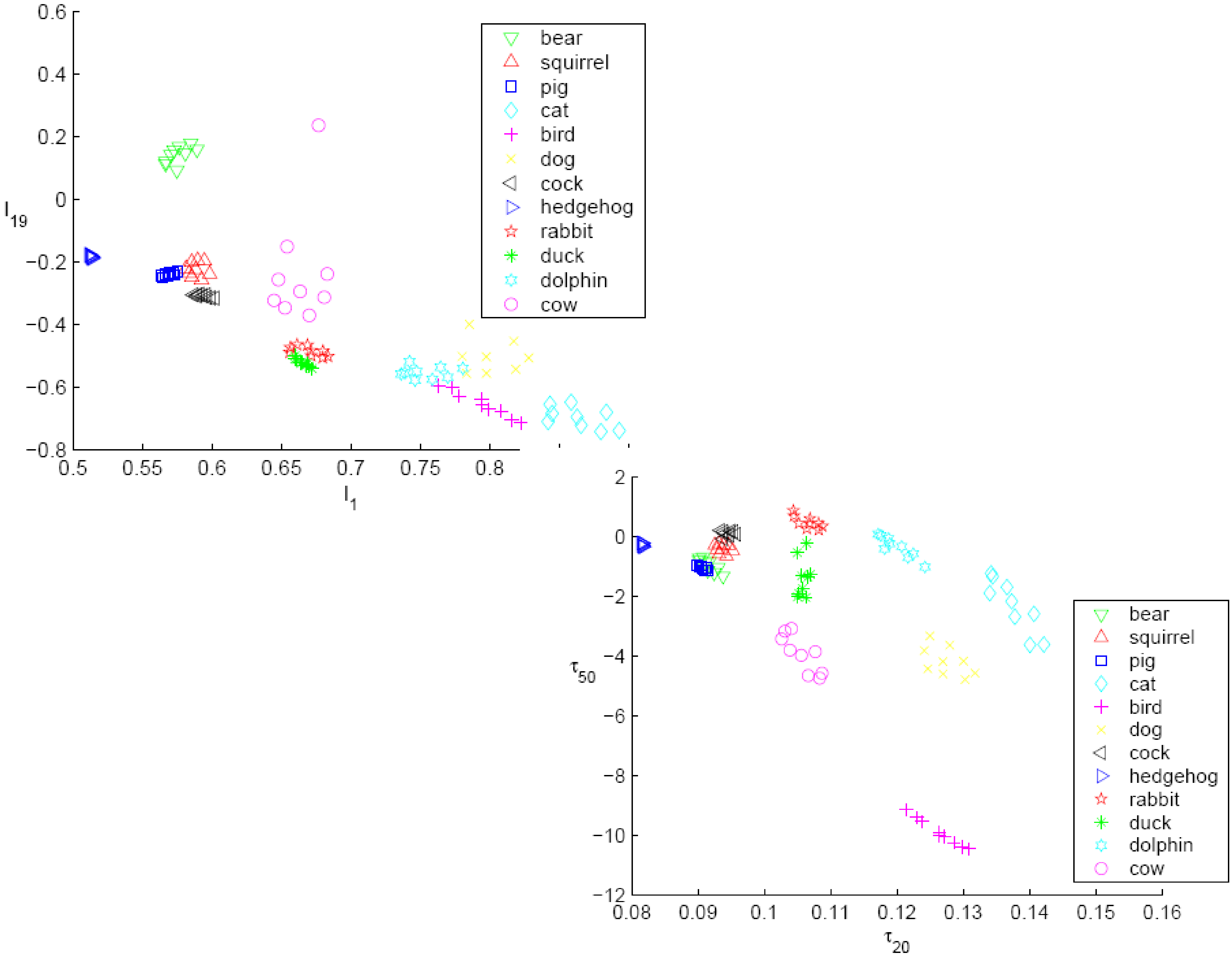




0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
I_{19}
0.5
0.52
0.54
0.56
0.58
I_1
equilateral triangle
isosceles triangle
square
rhombus
big rectangle
small rectangle
rhomboid
trapezoid
pentagon
regular hexagon
irregular hexagon
circle
ellipse

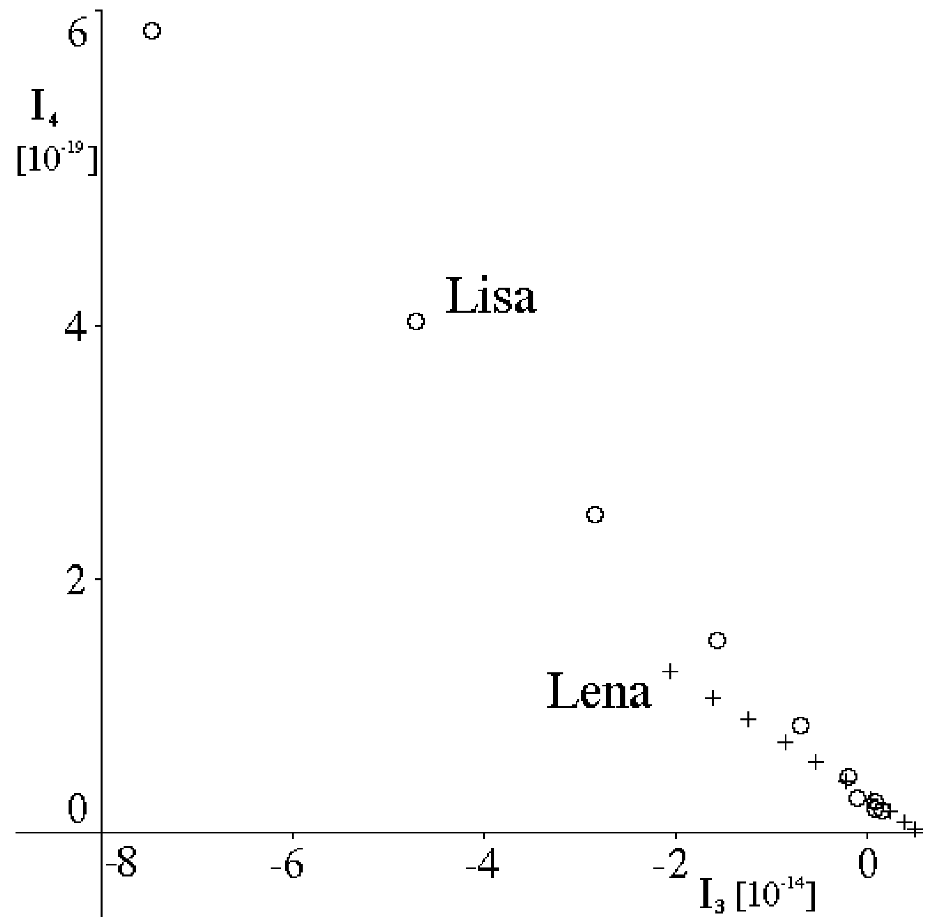


0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.2
−1.4
−1.6
−1.8
τ_{50}
0.075
0.08
0.085
0.09
0.095
0.1
0.105
τ_{20}
equilateral triangle
isosceles triangle
square
rhombus
big rectangle
small rectangle
rhomboid
trapezoid
pentagon
regular hexagon
irregular hexagon
circle
ellipse

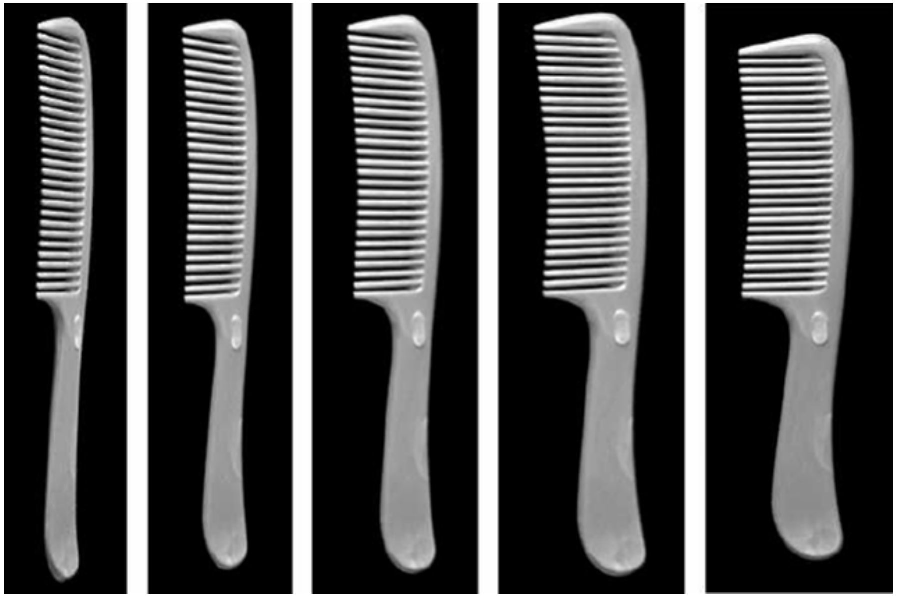


0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
I_{19}
0.5
0.55
0.6
0.65
0.7
0.75
0.8
I_1
bear
squirrel
pig
cat
bird
dog
cock
hedgehog
rabbit
duck
dolphin
cow
2
0
−2
−4
−6
−8
−10
−12
τ_{50}
0.08
0.09
0.1
0.11
0.12
0.13
0.14
0.15
0.16
τ_{20}
bear
squirrel
pig
cat
bird
dog
cock
hedgehog
rabbit
duck
dolphin
cow

Robustness of the AMI's to distortions



Robustness of the AMI's to distortions



$\gamma[^\circ]$	$I_1[10^{-2}]$	$I_2[10^{-6}]$	$I_3[10^{-4}]$	$I_4[10^{-5}]$	$I_6[10^{-3}]$	$I_7[10^{-5}]$	$I_8[10^{-3}]$	$I_9[10^{-4}]$
0	2.802	1.538	-2.136	-1.261	4.903	6.498	4.526	1.826
30	2.679	1.292	-1.927	-1.109	4.466	5.619	4.130	1.589
45	2.689	1.328	-1.957	-1.129	4.494	5.684	4.158	1.605
60	2.701	1.398	-2.011	-1.162	4.528	5.769	4.193	1.626
75	2.816	1.155	-1.938	-1.292	5.033	6.484	4.597	1.868

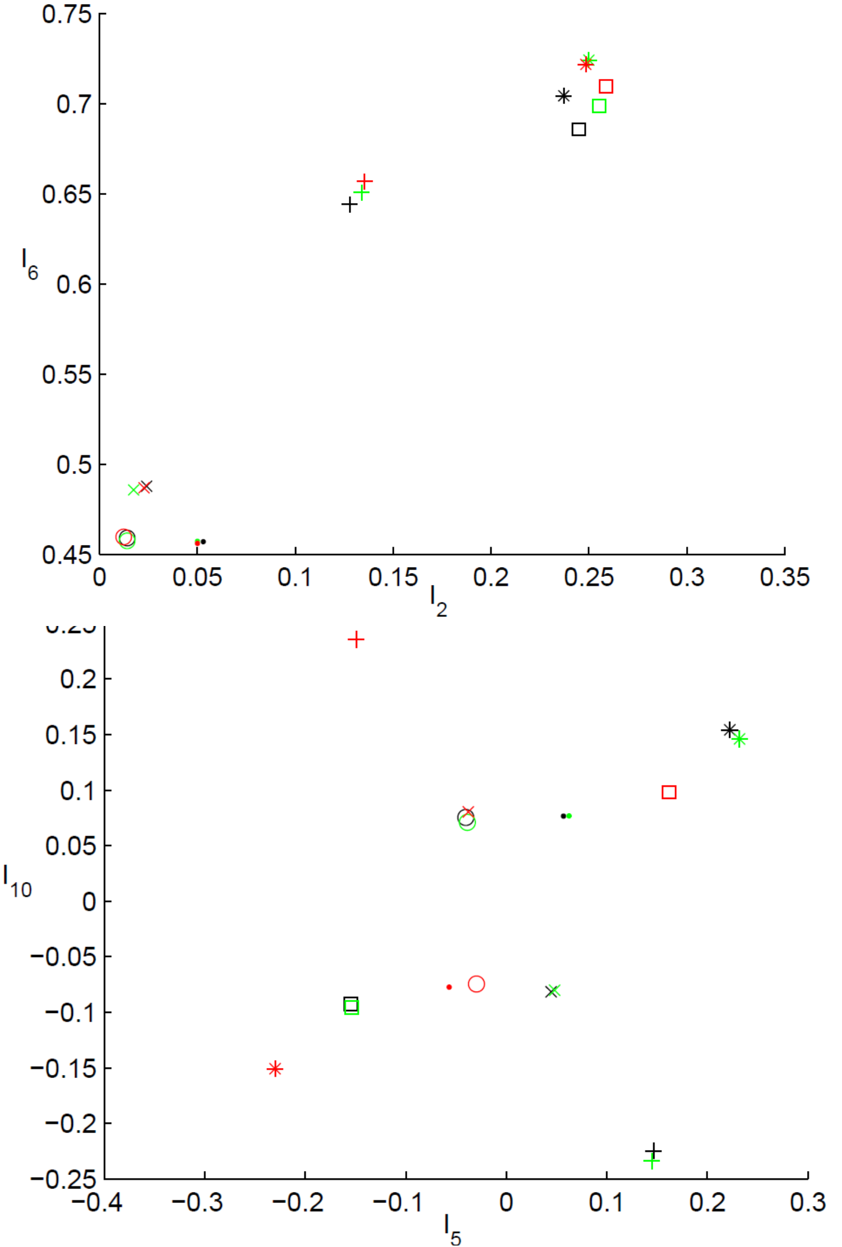
The role of pseudoinvariants

- They change the sign under mirroring (i.e. if $J < 0$)

- This may or may not be desirable depending on the application

- They are zero on all objects having axial symmetry

The role of pseudoinvariants



Affine invariants from Cayley-Aronhold equation

Skewing parameter *t*

$$\frac{dI}{dt} = \sum_p \sum_q \frac{\partial I}{\partial \mu_{pq}} \frac{d\mu_{pq}}{dt} = 0$$

$$\sum_p \sum_q p\mu_{p-1,q+1} \frac{\partial I}{\partial \mu_{pq}} = 0$$

$$I = \left(\sum_{j=1}^{n_t} c_j \prod_{\ell=1}^{r} \mu_{p_{j\ell},q_{j\ell}}\right) / \mu_{00}^{r+w}$$

Affine invariants from complex moments

$$I(c_{pq}) = (-2i)^w I(\mu_{pq})$$

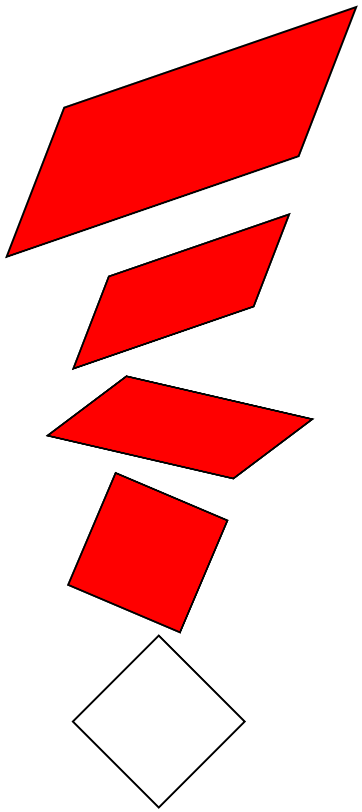
Affine invariants via normalization

Many possibilities how to define normalization constraints

Several possible decompositions of affine transform

Decomposition of the affine transform

- Horizontal and vertical translation
- Scaling
- First rotation
- Stretching
- Second rotation
- Mirror reflection



Normalization to partial transforms

- Horizontal and vertical translation -- $m_{01} = m_{10} = 0$
- Scaling -- $c_{00} = 1$
- First rotation -- c_{20} real and positive
- Stretching -- $c_{20} = 0$ $(\mu_{20}=\mu_{02})$
- Second rotation -- c_{21} real and positive

Moment values after the normalization

- Translation, uniform scaling and the first rotation

$$\mu'_{pq} = \sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j}(-1)^{p+j}\ \sin^{p-k+j}\alpha\ \cos^{q+k-j}\alpha\ \nu_{k+j,p+q-k-j}$$

$$\alpha = \frac{1}{2}\arctan\left(\frac{2\mu_{11}}{\mu_{20}-\mu_{02}}\right)$$

- Stretching

$$\mu''_{pq} = \delta^{p-q}\sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j}(-1)^{p+j}\ \sin^{p-k+j}\alpha\ \cos^{q+k-j}\alpha\ \nu_{k+j,p+q-k-j}$$

$$\delta = \sqrt[4]{\frac{\mu'_{02}}{\mu'_{20}}}$$

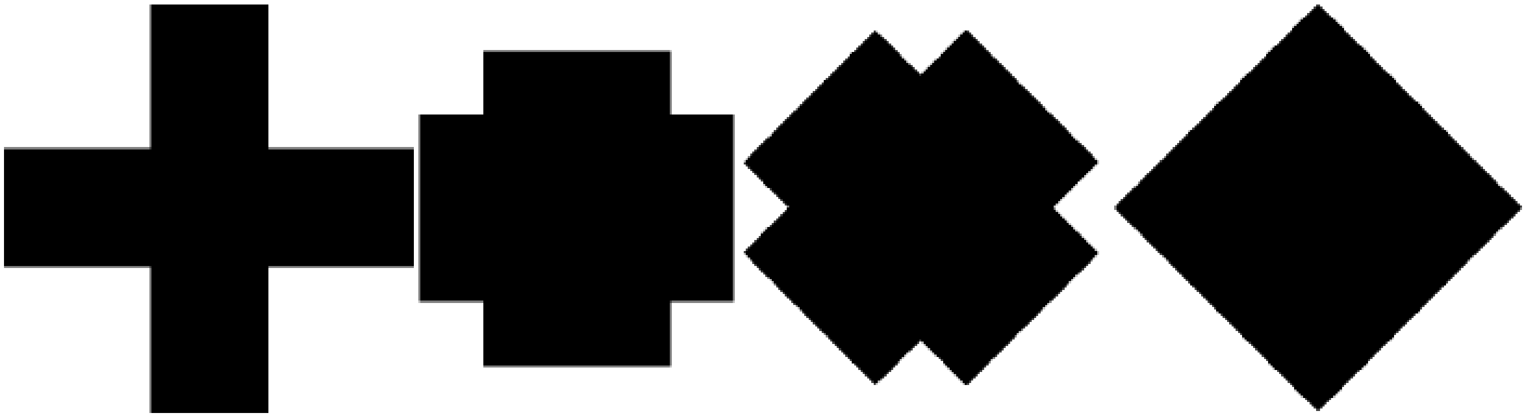
Moment values after the normalization

- Second rotation

$$\tau_{pq} = \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} (-1)^{p+j} \sin^{p-k+j} \varrho \cos^{q+k-j} \varrho \, \mu''_{k+j,p+q-k-j}$$

$$\varrho = \frac{1}{s-t} \arctan\left(\frac{\mathcal{I}m(c''_{st})}{\mathcal{R}e(c''_{st})} \right)$$

Possible volatility of the normalization



Affine invariants in 3D

3D affine transform

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b},$$

$$\mathbf{x}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \mathbf{b} = \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}$$

Analogy with the graph method

$$I(f) = \int_{-\infty}^{\infty} \prod_{j,k,\ell=1}^{r} C_{jk\ell}^{n_{jk\ell}} \cdot \prod_{i=1}^{r} f(x_i, y_i, z_i) dx_i dy_i dz_i,$$

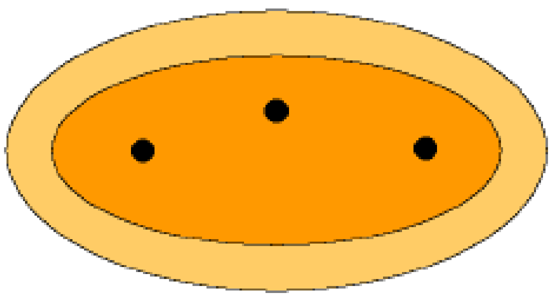
$$I(f)' = J^w |J|^r I(f)$$

Affine invariants in 3D

An example

$$I_1^{3D} = \frac{1}{6} \int_{-\infty}^{\infty} C_{123}^2 f(x_1,y_1,z_1) f(x_2,y_2,z_2) f(x_3,y_3,z_3) dx_1 dy_1 dz_1 dx_2 dy_2 dz_2 dx_3 dy_3 dz_3 / \mu_{000}^5$$
$$= (\mu_{200}\mu_{020}\mu_{002} + 2\mu_{110}\mu_{101}\mu_{011} - \mu_{200}\mu_{011}^2 - \mu_{020}\mu_{101}^2 - \mu_{002}\mu_{110}^2)/\mu_{000}^5.$$

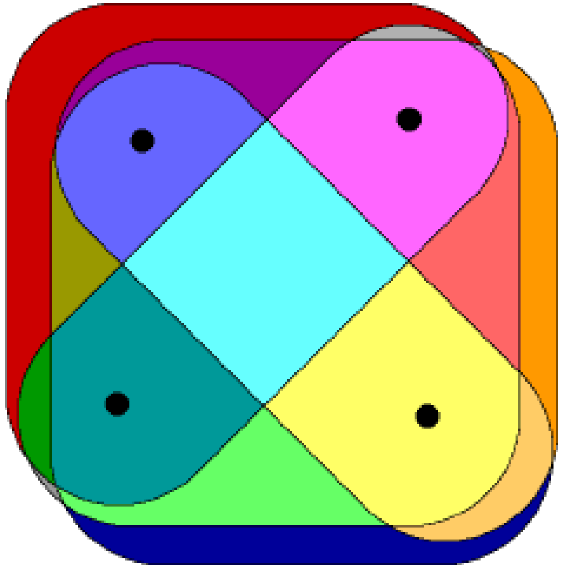
Corresponding hypergraph



Affine invariants in 3D

$$
\begin{aligned}
I_2^{3D} &= \frac{1}{36}\int_{-\infty}^{\infty} C_{123}C_{124}C_{134}C_{234} f(x_1,y_1,z_1)f(x_2,y_2,z_2)f(x_3,y_3,z_3)f(x_4,y_4,z_4) \\
&\quad dx_1dy_1dz_1dx_2dy_2dz_2dx_3dy_3dz_3dx_4dy_4dz_4/\mu_{000}^8 = \\
&= (\mu_{300}\mu_{003}\mu_{120}\mu_{021} + \mu_{300}\mu_{030}\mu_{102}\mu_{012} + \mu_{030}\mu_{003}\mu_{210}\mu_{201} - \mu_{300}\mu_{120}\mu_{012}^2 - \\
&\quad -\mu_{300}\mu_{102}\mu_{021}^2 - \mu_{030}\mu_{210}\mu_{102}^2 - \mu_{030}\mu_{201}^2\mu_{012} - \mu_{003}\mu_{210}^2\mu_{021} - \mu_{003}\mu_{201}\mu_{120}^2 - \\
&\quad -\mu_{300}\mu_{030}\mu_{003}\mu_{111} + \mu_{300}\mu_{021}\mu_{012}\mu_{111} + \mu_{030}\mu_{201}\mu_{102}\mu_{111} + \mu_{003}\mu_{210}\mu_{120}\mu_{111} + \\
&\quad +\mu_{210}^2\mu_{012}^2 + \mu_{201}^2\mu_{021}^2 + \mu_{120}^2\mu_{102}^2 - \mu_{210}\mu_{120}\mu_{102}\mu_{012} - \mu_{210}\mu_{201}\mu_{021}\mu_{012} - \\
&\quad -\mu_{201}\mu_{120}\mu_{102}\mu_{021} - 2\mu_{210}\mu_{012}\mu_{111}^2 - 2\mu_{201}\mu_{021}\mu_{111}^2 - 2\mu_{120}\mu_{102}\mu_{111}^2 + \\
&\quad +3\mu_{210}\mu_{102}\mu_{021}\mu_{111} + 3\mu_{201}\mu_{120}\mu_{012}\mu_{111} + \mu_{111}^4)/\mu_{000}^8.
\end{aligned}
$$

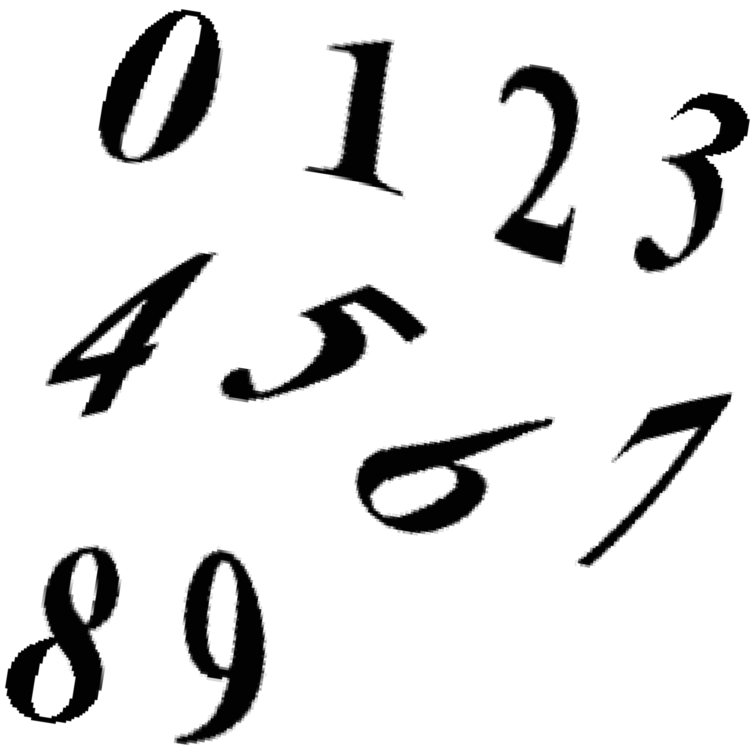
Corresponding hypergraph



Documented applications of the AMI’s

- Character/digit/symbol recognition in the case of skewing/slant
- Landmark recognition in robotics
- Recognition of aircraft and ships from non-perpendicular views
- Recognition of algae, fishes, whales, etc
- Image registration
- Target tracking
- Normalization of database images

Invariants to elastic deformations



How can we recognize objects on curved surfaces ...

Moment matching

- Find the best possible fit by minimizing the error and set

$$I(f, f') = \min_a \|\mathbf{m}' - A \cdot \mathbf{m}\|$$