Variational Methods in Image Processing

ÚTIA AV ČR

Outline

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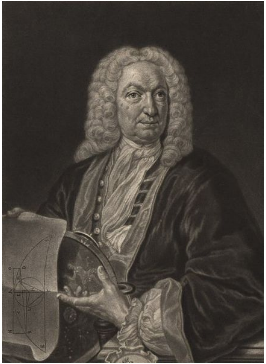
1. Introduction
   - Motivation
   - Derivation of Euler-Lagrange Equation
   - Variational Problem and P.D.E.

History

The Brachistochrone Problem:

"Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time."
Johann Bernoulli in 1696



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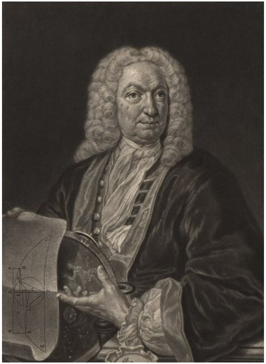

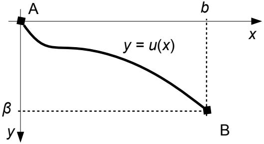

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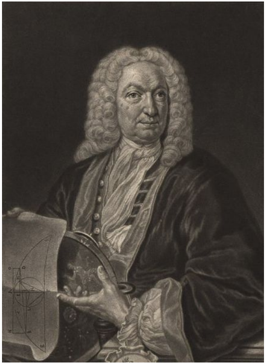
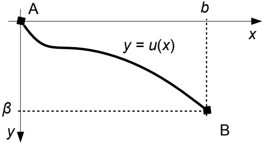
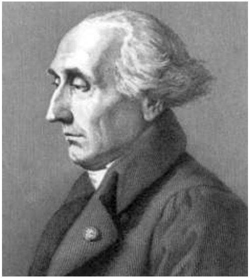
In one year Newton, Johann and Jacob Bernoulli, Leibniz, and de L'Hôpital came with the solution.





History

The problem was generalized and an analytic method was given by Euler (1744) and Lagrange (1760).



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- solution by means of Euler-Lagrange (E-L) equation

Calculus of Variations

Integral functionals

$$F(u) = \int_\Omega f(x, u(x), \nabla u(x))dx$$

Example

- $x \in R^2$ ... space of coordinates $[x_1, x_2]$
- Ω ... image support
- $u(x) : R^2 \rightarrow R$ ... grayscale image
- $\nabla u(x)$ ... image gradient $[u_{x_1}, u_{x_2}]$

Examples

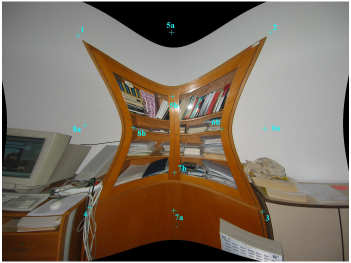
- Image Registration
  given a set of CP pairs $[x_i, y_i] \leftrightarrow [\tilde{x}_i, \tilde{y}_i]$
  find $\tilde{x} = f(x, y)$, $\tilde{y} = g(x, y)$

$$F(f) = \sum_i (\tilde{x}_i - f(x_i, y_i))^2 + \lambda \int \int f_{xx}^2 + 2f_{xy}^2 + f_{yy}^2 dxdy$$

  and a similar equation for $g(x, y)$

Examples

- Image Registration



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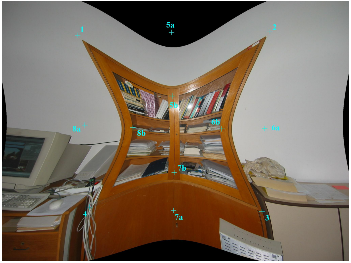
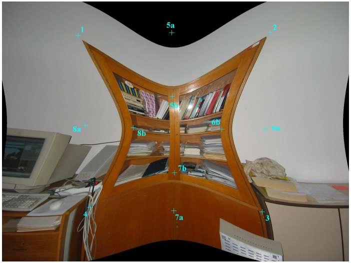


- Image Reconstruction
  given an image acquisition model $H(\cdot)$ and measurement g find the original image u

$$F(u) = \int (H(u) - g)^2 dx + \lambda \int |\nabla u|^2$$

Examples

- Image Registration



- Image Reconstruction

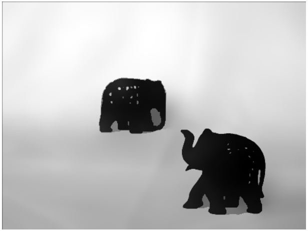
Examples

- Image Segmentation
  find a piece-wise constant representation u of an image g

$$F(u,K) = \int_{\Omega - K} (u-g)^2 dx + \alpha \int_{\Omega - K} |\nabla u|^2 dx + \beta \int_K ds$$

Examples

- Image Segmentation



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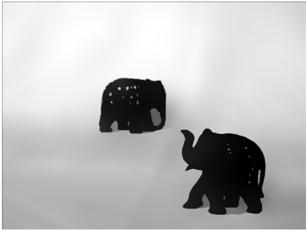
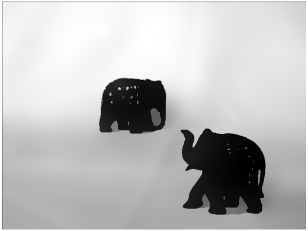
 

- Motion Estimation
  find velocity field $v(x) \equiv [v_1(x), v_2(x)]$ in an image sequence $u(x, t)$

$$F(v) = \int |v \cdot \nabla u + u_t| dx + \alpha \sum_j \int |\nabla v_j| dx + \beta \int c(\nabla u)|v|^2 dx$$

Examples

- Image Segmentation



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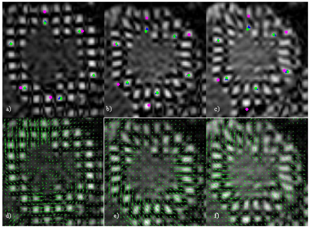

Examples

- Image classification

Examples

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- and many more

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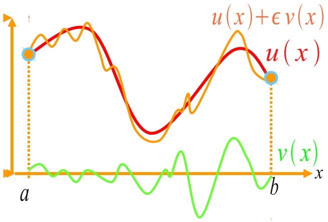
in 1-D ($g : R \to R$) we get the classical condition

$$g'(x) = 0$$

Variation of Functional

$$F(u) = \int_a^b f(x, u, u')dx$$



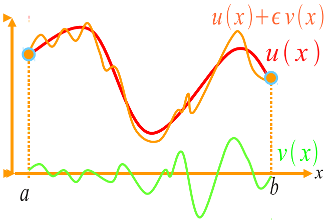
Variation of Functional

$$F(u) = \int_a^b f(x, u, u')dx$$

if u is extremum of F then from differential calculus follows

$$\frac{d}{d\varepsilon} F(u + \varepsilon v)\Big|_{\epsilon=0} = 0 \quad \forall v$$

$$F(u+\varepsilon v) = \int_a^b f(x, u+\varepsilon v, u' + \varepsilon v')dx$$



Partial derivatives

Example

$$f(x, u) = xu$$

$$\frac{\partial f}{\partial x} = u$$

$$\frac{df}{dx} = u$$

Partial derivatives

Example

$$f(x,u) = xu = xu(x) = x\sin x$$

$$\frac{\partial f}{\partial x} = u = \sin x$$

but

$$\frac{df}{dx} = \text{ chain rule} = \sin x + x\cos x$$

Chain Rule

$$\frac{d}{dx} f(u(x), v(x)) = \left(\frac{\partial}{\partial u} f(u, v)\right) \frac{du}{dx} + \left(\frac{\partial}{\partial v} f(u, v)\right) \frac{dv}{dx}$$

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Example

$$u(x) = x,\ v(x) = \sin x,\ f = uv = x \sin x$$

$$\frac{d}{dx} f(u, v) = v(x)1 + u(x) \cos x = \sin x + x \cos x$$

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per partes

$$\int_a^b uv' = uv\Big|_a^b - \int_a^b u'v$$

Derivation of E-L equation

$$\frac{d}{d\varepsilon}F(u+\varepsilon v) = \frac{d}{d\varepsilon}\int_a^b f(x, u+\varepsilon v, u' + \varepsilon v')$$

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chain rule

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$$= \int_a^b \frac{\partial f}{\partial u} v + \frac{\partial f}{\partial u'} v' \qquad \text{chain rule}$$

$$= \int_a^b \frac{\partial f}{\partial u} v - \int_a^b \frac{d}{dx}\frac{\partial f}{\partial u'} v + \frac{\partial f}{\partial u'} v \Big|_a^b \qquad \text{per partes}$$

Derivation of E-L equation

$$
\begin{aligned}
\frac{d}{d\varepsilon}F(u+\varepsilon v) &= \frac{d}{d\varepsilon}\int_a^b f(x, u+\varepsilon v, u'+\varepsilon v') \\
&= \int_a^b \frac{\partial f}{\partial u}v + \frac{\partial f}{\partial u'}v' && \text{chain rule} \\
&= \int_a^b \frac{\partial f}{\partial u}v - \int_a^b \frac{d}{dx}\frac{\partial f}{\partial u'}v + \frac{\partial f}{\partial u'}v\Big|_a^b && \text{per partes} \\
&= \int_a^b \left[\frac{\partial f}{\partial u} - \frac{d}{dx}\frac{\partial f}{\partial u'}\right]v + \frac{\partial f}{\partial u'}v\Big|_a^b = 0
\end{aligned}
$$

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$$= \int_a^b \left[\frac{\partial f}{\partial u} - \frac{d}{dx}\frac{\partial f}{\partial u'}\right] v + \frac{\partial f}{\partial u'}v\Big|_a^b = 0$$

to be equal to 0 for any v, $\left[\frac{\partial f}{\partial u} - \frac{d}{dx}\frac{\partial f}{\partial u'}\right] = 0 \rightarrow$ E-L equation

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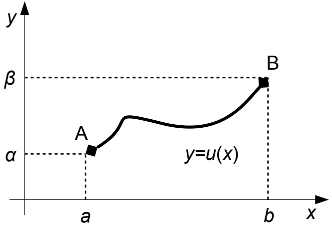
&= \int_a^b \frac{\partial f}{\partial u} v - \int_a^b \frac{d}{dx}\frac{\partial f}{\partial u'} v + \frac{\partial f}{\partial u'} v \Big|_a^b && \text{per partes} \\
&= \int_a^b \left[\frac{\partial f}{\partial u} - \frac{d}{dx}\frac{\partial f}{\partial u'}\right] v + \frac{\partial f}{\partial u'} v \Big|_a^b = 0
\end{aligned}
$$

to be equal to 0, we need boundary conditions,
e.g., fixed $u(a), u(b) \rightarrow v(a) = v(b) = 0$.

Toy case

Shortest path

- Find the shortest path between points A and B, assuming that one can write $y = u(x)$.



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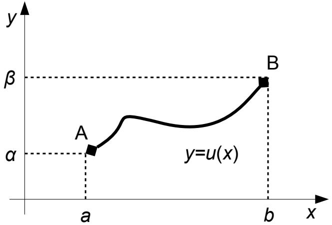

- We want to minimize $F(u(x)) = \int_a^b \sqrt{1 + u'(x)^2} dx$ with b.c. $u(a) = \alpha$, $u(b) = \beta$.

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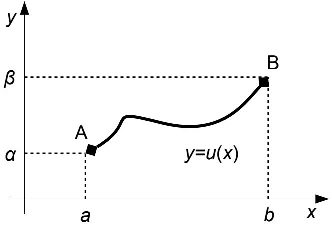

- We want to minimize $F(u(x)) = \int_a^b \sqrt{1 + u'(x)^2}dx$ with b.c. $u(a) = \alpha$, $u(b) = \beta$.
- E-L eq.: $-\frac{d}{dx}\frac{u'(x)}{\sqrt{1+u'(x)^2}} = 0$

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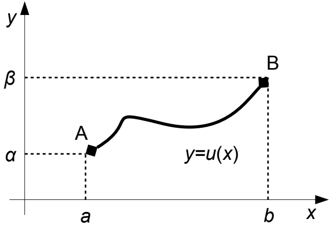

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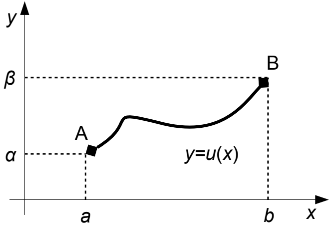

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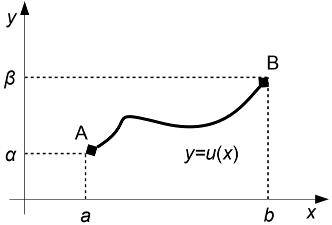

- We want to minimize $F(u(x)) = \int_a^b \sqrt{1 + u'(x)^2} dx$ with b.c. $u(a) = \alpha$, $u(b) = \beta$.
- E-L eq.: $-\frac{d}{dx}\frac{u'(x)}{\sqrt{1+u'(x)^2}} = 0 \Rightarrow u' = C\sqrt{1 + u'^2} \Rightarrow$ $u' =$ constant
- u(x) is a straight line between A and B.

E-L equation

If $u(x) : R^N \to R$ is extremum of $F(u) = \int_\Omega f(x, u, \nabla u) dx$,
where $\nabla u \equiv [u_{x_1}, \ldots, u_{x_N}]$
then

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then

$$F'(u) = \frac{\partial f}{\partial u}(x, u, \nabla u) - \sum_{i=1}^{N} \frac{d}{dx_i} \left(\frac{\partial f}{\partial u_{x_i}}(x, u, \nabla u) \right) = 0\,,$$

which is the E-L equation.

Beltrami Identity

$f(x, u, u')$

$$\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right) = 0$$

Beltrami Identity

$$f(x, u, u')$$

$$\frac{df}{dx} = \frac{\partial f}{\partial u} u' + \frac{\partial f}{\partial u'} u'' + \frac{\partial f}{\partial x}$$

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$$\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right) = 0$$

$$u' \frac{\partial f}{\partial u} - u' \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right) = 0$$

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$$\frac{d}{dx}\left(f - u'\frac{\partial f}{\partial u'}\right) - \frac{\partial f}{\partial x} = 0$$

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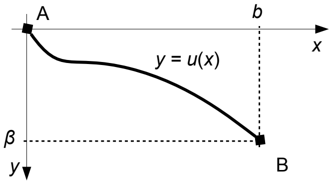
if $\frac{\partial f}{\partial x} = 0$ then

$$\frac{d}{dx}\left(f - u'\frac{\partial f}{\partial u'}\right) = 0 \Longleftrightarrow f - u'\frac{\partial f}{\partial u'} = C$$

Brachistochrone

- $F = \int dt$, $minF$ . . . curve of the shortest time.
- $F = \int \frac{ds}{v} = \int_0^b \frac{\sqrt{1+(u'(x))^2}}{v} dx$
- $\frac{1}{2}mv^2 = mgy(x) \Rightarrow v = \sqrt{2gu(x)}$



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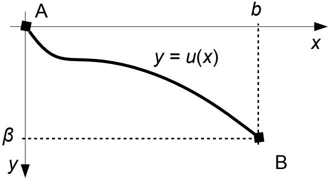
- $\frac{1}{2}mv^2 = mgy(x) \Rightarrow v = \sqrt{2gu(x)}$
- $$F = \int_0^b \frac{\sqrt{1+(u')^2}}{\sqrt{2gu}} dx$$



Brachistochrone

$$f(u, u') = \frac{\sqrt{1 + (u')^2}}{\sqrt{2gu}}$$

Brachistochrone

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Brachistochrone

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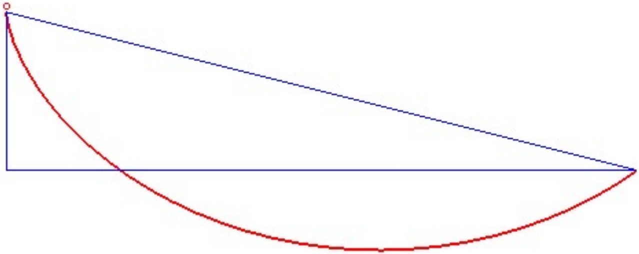
The solution $y = u(x)$ is a cycloid:

$$x(\theta) = \frac{1}{2}k(\theta - \sin\theta), \quad y(\theta) = \frac{1}{2}k(1 - \cos\theta)$$

Cycloid

Play avi



Play avi

Boundary conditions

- using "per partes" on $u(x,y)$, $\mathbf{n}(x,y) \equiv [n_1(x,y), n_2(x,y)]$ normal vector at the boundary $\partial\Omega$

$$\frac{\partial}{\partial\varepsilon} F(u+\varepsilon v) = \int (\cdot) dxdy + \int_{\partial\Omega} \left[\frac{\partial f}{\partial u_x} n_1 + \frac{\partial f}{\partial u_y} n_2 \right] v\, ds$$

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E-L equation example

- Smoothing functional:

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Outline

Steepest Descent

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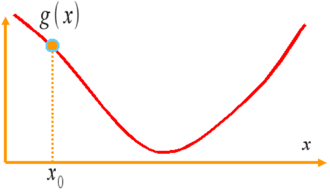

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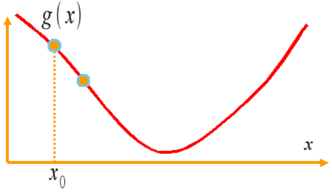

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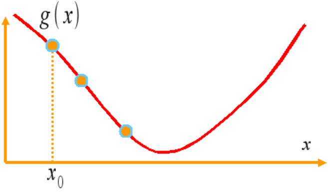

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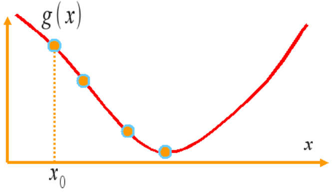

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- Finding the solution with the steepest-descent method is equivalent to solving P.D.E.:

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P.D.E - Gradient flow

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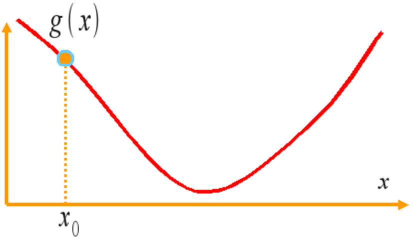
- Solving the variational problem with the steepest-descent method is equivalent to solving P.D.E.:

$$\frac{\partial u}{\partial t} = -F'(u)$$

+boundary conditions.

Steepest descent



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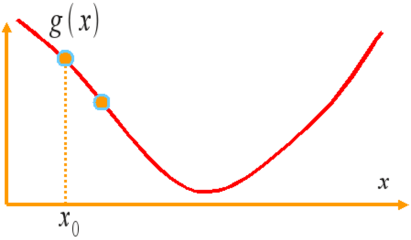

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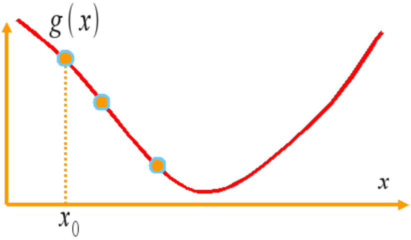

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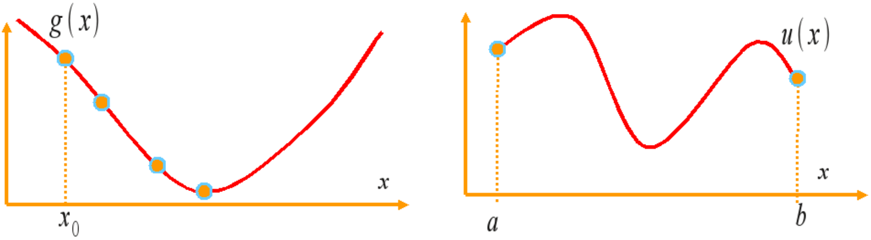

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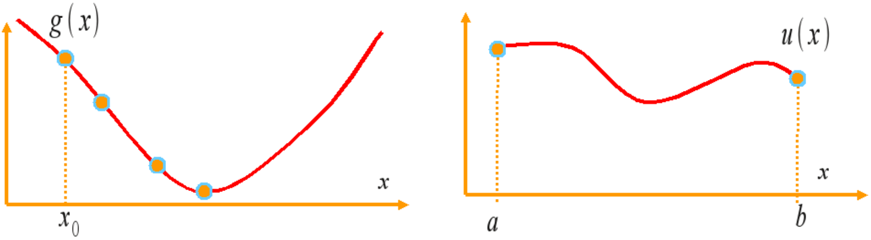

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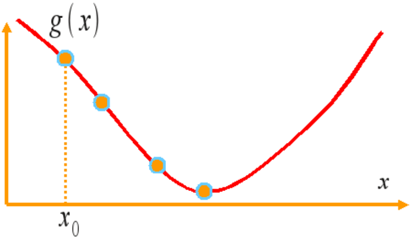

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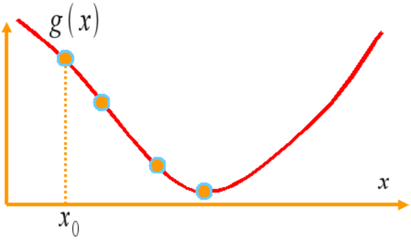

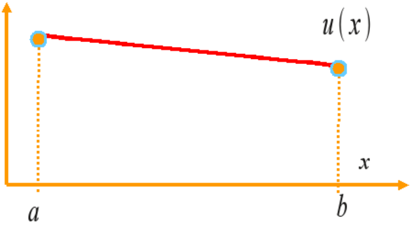

Differential Calculus x Variational Calculus

	Differential Calculus	Variational Calculus
Problem Spec.	function	function of function = functional
Necess. Cond.	1st derivative = 0	1st variation = 0
Result	one number (or vector)	function

Optimization Problem

- Solving PDE's is equivalent to optimization of integral functionals

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- Does every PDE have its corresponding optimization problem?

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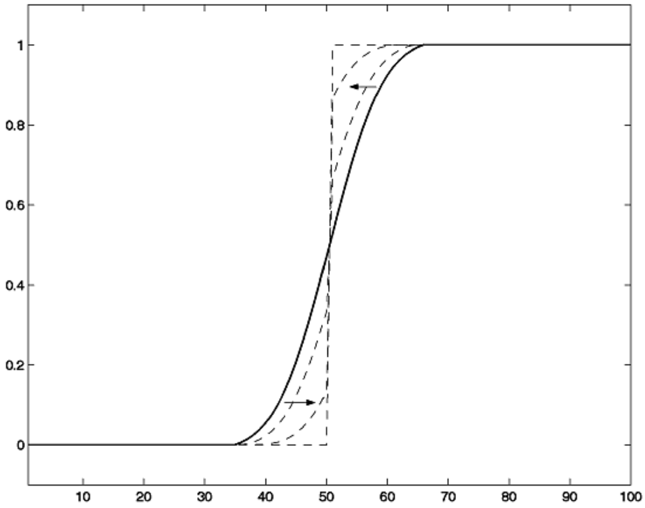
Example

$$u_t = \Delta u \quad \Leftrightarrow \quad \min \int_\Omega |\nabla u|^2$$

- Does every PDE have its corresponding optimization problem?
- Think of "shock filter": $u_t + \mathrm{sign}(\Delta u)\|\nabla u\| = 0$


1
0.8
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20
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70
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90
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